The Attraction of a Homogeneous Spherical Segment.

By G. Greenhill.

Mr. G. W. Hill has arrived at the unexpected result that the attraction of the homogeneous segment of a sphere (a flat lens) can be made to depend on the complete elliptic integral, first, second, and third, and thence is expressible by the function $F\phi$ and $E\phi$ of Legendre's Table IX (American Journal of Mathematics, Vol. XXIX, No. 4).

The result is unexpected, as the ordinary method of dissection leads to an integral of intractable nature, where elements of the integral are symmetrical with respect to the axis CO of the segment in Fig. 1.

But by cutting the segment up into slices by planes perpendicular to CP, drawn from C, the centre of the sphere, to the attracted point P, each slice QRQ' (Fig. 2) is the segment of a circle of which CP is the axis (the line through the centre of a circle perpendicular to its plane) and the components of attraction perpendicular and parallel to CP, as well as the potential of the segment at P, are given by simple functions.

This is the dissection employed by Mr. G. W. Hill; and a final integration by parts enables him to reduce the components of attraction of the spherical segment to an algebraical quadrature, which proves to be of elliptic character.

The object of this memoir is to resume the consideration of this elliptic integral and to show that the result can be made to depend on the complete elliptic integral of the first and second kind, and on two complete integrals of the third kind, expressible by incomplete integrals of the first and second kind.

Interpreted geometrically these two third elliptic integrals can be taken to represent the apparent area, or magnetic potential, of the base AQB (Fig. 3) of the segment, as seen from P, and another point P' on the radius CP which is inverse to P with respect to the spherical surface.

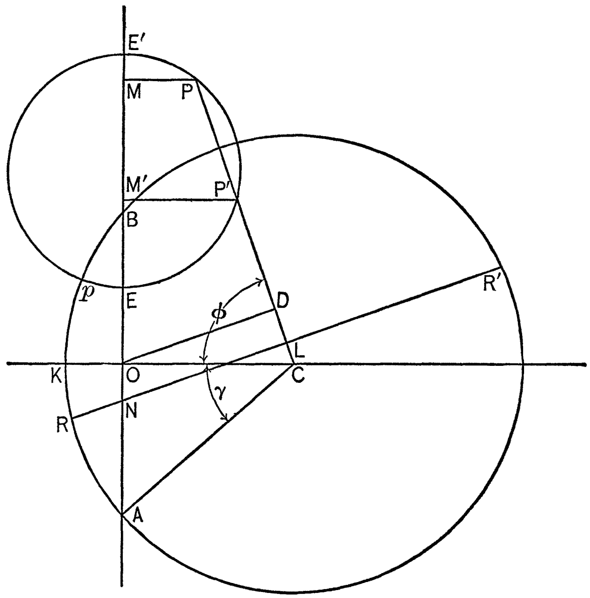

FIG. 1.

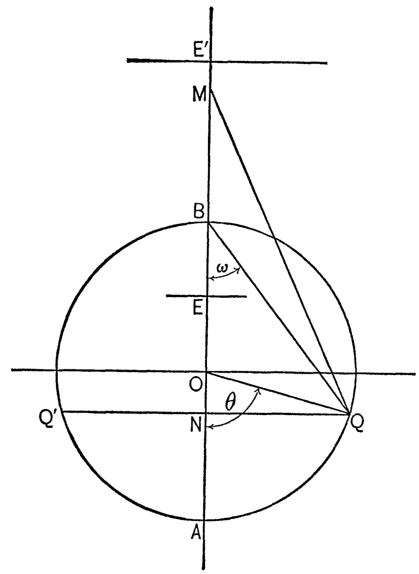

FIG. 3.

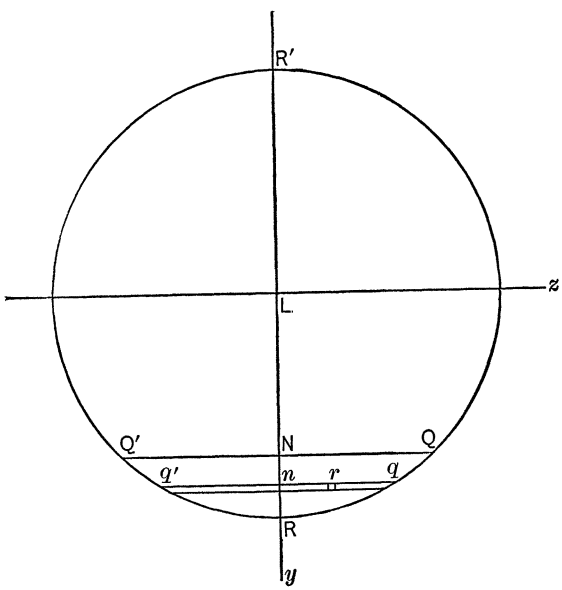

FIG. 2.

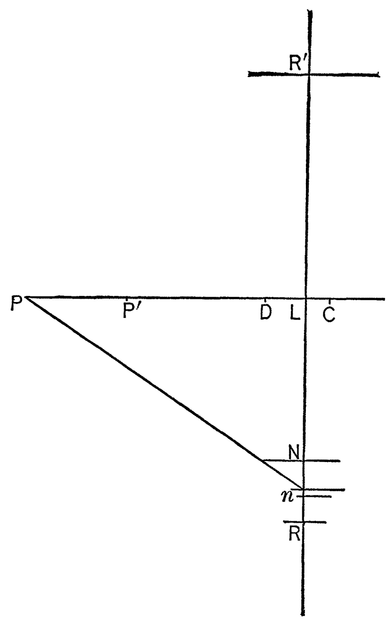

FIG. 4.

The analytical reduction is thus similar to that employed in the "Elliptic Integral in Electromagnetic Theory," *Transactions American Mathematical Society*, October, 1907, referred to in the sequel as *Trans. A M. S.*; and to facilitate the identification of results the same notation will be employed, taken from Maxwell's *E. and M.* (*Electricity and Magnetism*).

Used in conjunction with Mr. G. W. Hill's notation, this will require a change of his a into c to represent the radius of the sphere, retaining Maxwell's a to represent the radius of the base of the segment.

As we shall not require Mr. Hill's b, no change need be made to avoid confusion with Maxwell's b, representing MP, the distance of P from the plane of the base AB.

The other symbols, x, γ, ϕ, are retained, as employed by Mr. Hill; but we have found it convenient to change his x' into r, although we shall have occasion to use r to denote PQ.

There are so many quantities to be distinguished that the same letter must be used occasionally, with due warning, for more than one meaning, if the conventional notation is to be preserved.

Following Mr. Hill, we begin by the calculation of Y, the component of the attraction perpendicular to CP, by slicing the spherical segment into segments of a circle on the same axis CP, made by planes perpendicular to CP; the component attraction of the circular segment perpendicular to CP then represents

$$\frac{1}{G\rho}\frac{dY}{dx}$$

in the spherical segment, ρ denoting the density and G the constant of gravitation.

Attraction of the Circular Segment QRQ' at a Point P on its Axis CP (Fig. 2 and 4).

1. The attraction at P in the direction Pn of the line element qq' is:

$$\int_{-nq}^{nq}\frac{Pn}{Pr}\cdot\frac{dz}{Pr^2}=\int_0^{nq}\frac{2\,Pn\,dz}{(Pn^2+z^2)^{\frac{3}{2}}}=\frac{nr}{Pr\,.\,Pn}\Big]_{-nq}^{nq}=\frac{2nq}{PQ\,.\,Pn}, \tag{1}$$

so that the component attraction of the segment QRQ' in the direction LR perpendicular to CP is:

$$\int_{LN}^{LR}\frac{Ln}{Pn}\cdot\frac{2\,nq\,dy}{PQ\,.\,Pn}=\frac{2}{PQ}\int\frac{\surd(LQ^2-y^2)\,y\,dy}{PL^2+y^2}$$

$$=2\,PQ\int\frac{y\,dy}{(PL^2+y^2)\surd(LQ^2-y^2)}-\frac{2}{PQ}\int\frac{y\,dy}{\surd(LQ^2-y^2)}$$

$$=2\,\mathrm{th}^{-1}\frac{NQ}{PQ}-2\,\frac{NQ}{PQ}=2\,J_1-2\,\frac{NQ}{PQ}, \tag{2}$$

where

$$J_1 = \text{th}^{-1}\frac{NQ}{PQ} = \text{ch}^{-1}\frac{PQ}{PN} = \text{sh}^{-1}\frac{QN}{PN}, \tag{3}$$

or

$$J_1 = \tfrac{1}{2}\log\frac{PQ+NQ}{PQ-NQ} = \tfrac{1}{2}\log\frac{1+P}{1-P}, \tag{4}$$

in Mr. Hill's notation for P; and

$$\text{th}\, J_1 = \frac{NQ}{PQ} = P = \sin QPN, \tag{5}$$

and the angle QPN is the hyperbolic amplitude of J_1.

2. The notation, representing the various lines on the figures, is taken to agree as closely as possible with Maxwell and Mr. Hill:

$$\begin{gathered} OA = a, \qquad OM = A, \qquad MP = b, \\ OCP = \phi, \qquad OCA = \gamma, \qquad BA = c, \\ OA = c\sin\gamma = a, \qquad OC = c\cos\gamma = a\cot\gamma, \\ CP = r \text{ (or } x'), \qquad CL = x, \qquad LP = r - x, \\ MP = c\cos\gamma - r\cos\phi = b, \ldots\ldots \end{gathered}$$

Supposing x to range between

$$x_2 = \mathrm{a} = c\cos(\phi - \gamma) \text{ and } x_3 = \mathrm{b} = c\cos(\phi + \gamma), \tag{1}$$

$$x_2 - x \,.\, x - x_3 = -x^2 + 2cx\cos\phi\cos\gamma - c^2\cos^2\phi + c^2\sin^2\gamma. \tag{2}$$

To pass from x, the variable employed by Mr. Hill, to Maxwell's θ, representing the angle AOQ in the plane of the base in Fig. 3,

$$x = CD - ON\sin\phi = c\cos\phi\cos\gamma - a\sin\phi\cos\theta, \tag{3}$$

$$PQ^2 = r^2 + c^2 - 2rx = A^2 + 2Aa\cos\theta + a^2 + b^2, \tag{4}$$

$$\tfrac{1}{2}(PA^2 + PB^2) = r^2 - 2cr\cos\phi\cos\gamma + c^2 = A^2 + a^2 + b^2, \tag{5}$$

$$LN = MN\cos\phi + PM\sin\phi = (A + a\cos\theta)\cos\phi + b\sin\phi, \tag{6}$$

$$PL = MN\sin\phi - PM\cos\phi = (A + a\cos\theta)\sin\phi - b\cos\phi, \tag{7}$$

$$PN^2 = (A + a\cos\theta)^2 + b^2, \ldots\ldots \tag{8}$$

Then

$$\begin{aligned} \frac{1}{2G\rho}\frac{dY}{d\theta} &= J_1 a\sin\phi\sin\theta - \frac{a\sin\theta}{PQ}\, a\sin\phi\sin\theta \\ &= -a\sin\phi\frac{d}{d\theta}(J_1\cos\theta) + a\sin\phi\cos\theta\frac{dJ_1}{d\theta} - \sin\phi\frac{QN^2}{PQ}. \end{aligned} \tag{9}$$

Now

$$\frac{dJ_1}{d\theta}=\frac{d}{d\theta}\,\mathrm{sh}^{-1}\frac{QN}{PQ}=\frac{QN}{PQ}\left(\frac{\cos\theta}{\sin\theta}+\frac{A+a\cos\theta}{PN^2}a\sin\theta\right)$$

$$=\frac{a\cos\theta}{PQ}+\frac{QN^2}{PN^2}\cdot\frac{A+a\cos\theta}{PQ}, \tag{10}$$

$$\frac{1}{2G\rho}\frac{dY}{d\theta}=-a\sin\phi\frac{d}{d\theta}(J_1\cos\theta)+\sin\phi\frac{a^2\cos^2\theta}{PQ}$$

$$+\sin\phi\frac{QN^2}{PN^2}\frac{(A+a\cos\theta)\,a\cos\theta}{PQ}-\sin\phi\frac{QN^2}{PQ}, \tag{11}$$

$$\frac{1}{2G\rho\sin\phi}\frac{dY}{d\theta}=$$

$$-a\frac{d}{d\theta}(J_1\cos\theta)+\frac{a^2\cos^2\theta}{PQ}+\frac{QN^2}{PN^2}\frac{(A+a\cos\theta)\,a\cos\theta}{PQ}-\frac{QN^2}{PQ}$$

$$=\quad\star\quad+\quad\star\quad+\frac{QN^2}{PN^2}\cdot\frac{PN^2-b^2-A(A+a\cos\theta)}{PQ}-\frac{QN^2}{PQ}$$

$$=\quad\star\quad+\quad\star\quad-\frac{QN^2}{PN^2}\cdot\frac{b^2}{PQ}-\left(A\frac{dJ_1}{d\theta}-\frac{Aa\cos\theta}{PQ}\right)$$

$$=-\frac{d}{d\theta}(A+a\cos\theta)J_1+\frac{a^2}{PQ}+\frac{Aa\cos\theta}{PQ}-\frac{QN^2}{PQ}-\frac{QN^2}{PN^2}\cdot\frac{b^2}{PQ}, \tag{12}$$

the star representing a term repeated.

Now J_1 vanishes at the limits, A and B, so that

$$\frac{Y}{G\rho\sin\phi}=\int_0^{2\pi}\frac{a^2\,d\theta}{PQ}+\int\frac{Aa\cos\theta\,d\theta}{PQ}-\int\frac{QN^2\,d\theta}{PQ}-b\int\frac{QN^2}{PN^2}\cdot\frac{b\,d\theta}{PQ}, \tag{13}$$

consisting of complete elliptic integrals, of the first and second kind, and the last term of the third kind, representing geometrically the apparent area Ω of the base AB as seen from P; as is proved by a dissection in Fig. 3 into line elements QNQ' of length $2a\sin\theta$ and breadth $a\sin\theta\,d\theta$; the apparent area $d\Omega$ at P of the element being

$$d\Omega=\frac{2a\sin\theta}{PQ}\cdot\frac{b}{PN}\cdot\frac{a\sin\theta\,d\theta}{PN}=2\frac{QN^2}{PN^2}\cdot\frac{b}{PQ}. \tag{14}$$

Thus

$$\frac{Y}{G\rho}=(Pa-QA-S-\Omega b)\sin\phi, \tag{15}$$

$$\left.\begin{aligned}&P=\int_0^{2\pi}\frac{a\,d\theta}{PQ},\quad Q=\int\frac{-a\cos\theta\,d\theta}{PQ},\quad S=\int\frac{QN^2\,d\theta}{PQ},\\&\qquad\Omega=\int\frac{QN^2}{PN^2}\cdot\frac{b\,d\theta}{PQ},\end{aligned}\right\} \tag{16}$$

and on reference to the *Trans. A. M. S.*, P is the potential of the circumference of the circular base AB, and $2\pi QA$ is Maxwell's M of § 701, *E. and M.*

3. The attraction at P of the line element QQ' of the circular base AB in Fig. 3 is along PN, and by (1), § 1, is equal to

$$\frac{QQ'}{PN.PQ}, \text{ having components } \frac{QQ'.PM}{PN^2.PQ} \text{ and } \frac{QQ'.MN}{PN^2.PQ}, \tag{1}$$

perpendicular and parallel to the base; so that, W denoting the potential at P of the base AB, covered with superficial density σ,

$$-\frac{1}{G\sigma}\frac{dW}{db} = \int_0^\pi \frac{QQ'.PM}{PN^2.PQ} a \sin\theta\, d\theta = 2\int \frac{QN^2}{PN^2} \cdot \frac{b\,d\theta}{PQ} = \Omega, \tag{2}$$

$$-\frac{1}{G\sigma}\frac{dW}{dA} = \int \frac{QQ'.MN}{PN^2.PQ} a \sin\theta\, d\theta = 2\int \frac{QN^2}{PN^2} \cdot \frac{A + a\cos\theta}{PQ}\, d\theta$$

$$= 2\int \left(\frac{dJ_1}{d\theta} - \frac{a\cos\theta}{PQ}\right) d\theta = Q, \tag{3}$$

since J_1 vanishes at both limits, A and B; also

$$\frac{1}{G\sigma}\frac{dW}{da} = P, \tag{4}$$

the potential of the circumference of the circle AB.

Then, since W is a homogeneous function of the first degree in a, A, b,

$$W = a\frac{dW}{da} + A\frac{dW}{dA} + b\frac{dW}{db}, \tag{5}$$

$$\frac{W}{G\sigma} = Pa - QA - \Omega b, \tag{6}$$

and we can write

$$\frac{Y}{G\rho} = \left(\frac{W}{G\sigma} - S\right)\sin\phi, \tag{7}$$

and it is shown in the sequel, (13), (14), § 13, that

$$S = -\tfrac{1}{3}\frac{PA^2.PB^2}{Ab}\frac{d\Omega}{dA} = \tfrac{2}{3}Pa - \tfrac{1}{3}Q\frac{A^2 + a^2 + b^2}{A} \tag{8}$$

Interpreted in electromagnetism, Ω is the magnetic potential of the base AB magnetized normally, or of unit current circulating round the rim, while Q is the potential when the circle is magnetized parallel to AB; and $2\pi QA$ is the vector potential on Stokes current function of Ω (*E. and M.*, § 703).

Along the axis CO, $\phi = 0$, $A = 0$, $PQ = PA$:

$$\Omega = 2\pi\left(1 - \frac{PO}{PA}\right), \tag{9}$$

$$\frac{Y}{G\rho\sin\phi} = 2\int_0^\pi \frac{a^2\,d\theta}{PA} + 2\int \frac{Aa\cos\theta\,d\theta}{PA}$$

$$- 2\int \frac{a^2\sin^2\theta\,d\theta}{PA} - 2\pi PO\left(1 - \frac{PO}{PA}\right)$$

$$= \frac{2\pi a^2}{PA} + 0 - \frac{\pi a^2}{PA} - 2\pi\,.\,PO + 2\pi\frac{PO^2}{PA} = \pi\frac{(PA - PO)^2}{PA}. \tag{10}$$

Thus a smooth particle on the surface will make $n/2\pi$ ~/sec., about the position of equilibrium, at the vertex K or at O, where

$$n^2 = G\rho\pi\frac{(KA - KO)^2}{KA\,.\,KC}, \text{ or } G\rho\pi\frac{OA}{OC}. \tag{11}$$

4. The component attraction of the segment QRQ' in the direction PC is

$$\int \frac{PL}{Pn}\cdot\frac{2nq\,dy}{PQ\,.\,Pn} = 2\frac{PL}{PQ}\int \frac{\sqrt{(LQ^2 - y^2)}\,dy}{PL^2 + y^2}$$

$$= 2\int \frac{PL\,.\,PQ\,.\,dy}{(PL^2 + y^2)\sqrt{(LQ^2 - y^2)}} - 2\frac{PL}{PQ}\int \frac{dy}{\sqrt{(LQ^2 - y^2)}}$$

$$= 2J_3 - 2\frac{PL}{PQ}J_2, \tag{1}$$

where

$$J_2 = \cos^{-1}\frac{LN}{LQ} = \sin^{-1}\frac{QN}{LQ} = \tan^{-1}\frac{QN}{LN}, \tag{2}$$

$$J_3 = \cos^{-1}\frac{PQ\,.\,LN}{PN\,.\,LQ} = \sin^{-1}\frac{PL\,.\,NQ}{PN\,.\,LQ} = \tan^{-1}\frac{PL\,.\,NQ}{PQ\,.\,LN}; \tag{3}$$

and J_2 is QLN, the angle between the planes PLQ, PLA; while J_3 is the complement of the angle between the planes PQN, PQL.

Then, in Mr. Hill's notation,

$$\frac{1}{2G\rho}\frac{dX'}{dx} = J_3 - \frac{PL}{PQ}J_2, \tag{4}$$

and X' will represent X, the component attraction along PC, provided PC does not cut the base AB; otherwise a modification is required which is given in the sequel. We divide X' into two parts, X_1 and X_2, where

$$\frac{1}{2G\rho}\frac{dX_1}{dx} = J_3, \qquad \frac{1}{2G\rho}\frac{dX_2}{dx} = -\frac{PL}{PQ}J_2; \tag{5}$$

and then

$$\frac{1}{2\,G\rho\sin\phi}\,\frac{dX_1}{d\theta} = a\,J_3\sin\theta = -\frac{d}{d\theta}\,(a\,J_3\cos\theta) + a\cos\theta\,\frac{dJ_3}{d\theta}, \tag{6}$$

and, with $LQ^2 = c^2 - x^2$,

$$\frac{dJ_3}{d\theta} = \frac{d}{d\theta}\sin^{-1}\frac{PL\,.\,NQ}{PN\,.\,LQ}$$

$$= \frac{PL\,.\,NQ}{PQ\,.\,LN}\left[\frac{-a\sin\theta\sin\phi}{PL} + \frac{(A + a\cos\theta)\,a\sin\theta}{PN^2} + \frac{\cos\theta}{\sin\theta} + \frac{x\dfrac{dx}{d\theta}}{LQ^2}\right], \tag{7}$$

in which, with

$$PL = (A + a\cos\theta)\sin\phi - b\cos\phi, \quad LN = (A + a\cos\theta)\cos\phi + b\sin\phi, \tag{8}$$

$$\frac{-a\sin\phi\sin\theta}{PL} + \frac{(A + a\cos\theta)\,a\sin\theta}{PN^2} = -b\,\frac{LN\,.\,a\sin\theta}{PL\,.\,PN^2}, \tag{9}$$

$$\frac{\cos\theta}{\sin\theta} + \frac{x\dfrac{dx}{d\theta}}{LQ^2} = \frac{LN\,[a\cos\phi + (A\cos\phi + b\sin\phi)\cos\theta]}{\sin\theta\,LQ^2}$$

$$= \frac{LN\,(c\cos\phi - x\cos\gamma)}{\sin\gamma\sin\theta\,LQ^2}, \tag{10}$$

so that

$$\frac{dJ_3}{d\theta} = -\frac{a^2\sin^2\theta}{LN^2}\cdot\frac{b}{PQ} + \frac{a}{\sin\gamma}\,\frac{PL\,(c\cos\phi - x\cos\gamma)}{LQ^2\,.\,PQ}$$

$$= -\tfrac{1}{2}\,\frac{d\Omega}{d\theta} + c\,\frac{(r - x)\,(c\cos\phi - x\cos\gamma)}{(c^2 - x^2)\,PQ} \tag{11}$$

and, with

$$a\sin\phi\cos\theta = c\cos\phi\cos\gamma - x, \tag{12}$$

$$\frac{1}{2\,G\rho}\,\frac{dX_1}{d\theta} =$$

$$-\sin\phi\,\frac{d}{d\theta}(a\,J_3\cos\theta) + a\sin\phi\cos\theta\left[-\frac{QN^2}{PN^2}\cdot\frac{b}{PQ} + c\,\frac{(r-x)(c\cos\phi - x\cos\gamma)}{LQ^2\,.\,PQ}\right]$$

$$= \quad\star \quad + \sin\phi\left(-b\,\frac{dJ_1}{d\theta} + \frac{ab\cos\theta}{PQ} + \tfrac{1}{2}A\,\frac{d\Omega}{d\theta}\right)$$

$$+ c\,\frac{(c\cos\phi\cos\gamma - x)\,(r - x)}{c^2 - x^2}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \star + \star - c\,\frac{c\cos\phi - x\cos\gamma}{PQ} + \frac{\tfrac{1}{2}c\,(1 + \cos\phi\cos\gamma)\,(r + c)}{c + x}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$- \frac{\tfrac{1}{2}c\,(1 - \cos\phi\cos\gamma)\,(r - c)}{c - x}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}, \tag{13}$$

so that three elliptic integrals of the third kind appear in this expression for X_1.

But if P' is the point on CP inverse to P with respect to the spherical surface

$$CP' = \frac{c^2}{r}, \quad P'L = \frac{c^2}{r} - x, \quad \text{and} \quad \frac{P'Q}{PQ} = \frac{c}{r}, \tag{14}$$

and with

$$J_3' = \cos^{-1}\frac{P'Q\,.\,LN}{P'N\,.\,LQ} = \sin^{-1}\frac{P'L\,.\,NQ}{P'N\,.\,LQ}, \tag{15}$$

$$\begin{aligned}\frac{dJ_3'}{d\theta} &= -\tfrac{1}{2}\frac{d\Omega'}{d\theta} + c\,\frac{\left(\frac{c^2}{r} - x\right)(c\cos\phi - x\cos\gamma)}{LQ^2\,.\,P'Q} \\ &= -\tfrac{1}{2}\frac{d\Omega'}{d\theta} + \frac{(c^2 - xr)(c\cos\phi - x\cos\gamma)}{(c^2 - x^2)\,PQ},\end{aligned} \tag{16}$$

where Ω' denotes the apparent area of the base AB at P'; and then

$$\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega') = \frac{r+c}{c+x}\,.\,\frac{c\cos\phi - x\cos\gamma}{PQ}, \tag{17}$$

$$\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega') = \frac{r-c}{c-x}\,.\,\frac{c\cos\phi - x\cos\gamma}{PQ}, \tag{18}$$

$$\begin{aligned}\frac{1}{2G\rho}\frac{dX_1}{d\theta} = \\ -\sin\phi\frac{d}{d\theta}(aJ_3\cos\theta) - b\sin\phi\frac{dJ_1}{d\theta} + \sin\phi\frac{ab\cos\theta}{PQ} - c\,\frac{c\cos\phi - x\cos\gamma}{PQ} \\ + \tfrac{1}{2}(b\cos\phi + r)\frac{d\Omega}{d\theta} + \tfrac{1}{2}c\frac{d\Omega'}{d\theta} + c\cos\phi\cos\gamma\frac{dJ_3}{d\theta} + c\frac{dJ_3'}{d\theta}.\end{aligned} \tag{19}$$

For the calculation of X_2,

$$\frac{1}{2G\rho}\frac{dX_2}{dx} = -\frac{PL}{PQ}J_2, \tag{20}$$

and

$$\int\frac{PL}{PQ}dx = \int\frac{r-x}{(r^2+c^2-2rx)^{\frac{1}{2}}}dx = \frac{rx - 2r^2 + c^2}{3r^2}PQ, \tag{21}$$

so that

$$\frac{1}{2G\rho}\frac{dX_2}{d\theta} = -\frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2}\,.\,PQ\,.\,J_2\right) + \frac{rx - 2r^2 + c^2}{3r^2}PQ\frac{dJ_2}{d\theta}, \tag{22}$$

where

$$\left.\begin{aligned} J_2 &= \tan^{-1}\frac{QN}{LN} = \tan^{-1}\frac{a\sin\theta}{(a\cos\theta + A)\cos\phi + b\sin\phi}, \\ \frac{dJ_2}{d\theta} &= a\,\frac{(A\cos\theta + c)\cos\phi + b\cos\theta\sin\phi}{LQ^2} = c\,\frac{c\cos\phi - x\cos\gamma}{LQ^2}. \end{aligned}\right\} \quad (23)$$

$$\frac{1}{2G\rho}\frac{dX_2}{d\theta} = -\frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2}\,.\,PQ\,.\,J_2\right)$$

$$+\frac{c}{3r^2}\,.\,\frac{(rx - 2r^2 + c^2)(-2rx + r^2 + c^2)}{c^2 - x^2}\,.\,\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \quad \star \quad + \frac{c}{3r^2}\left[2r^2 - \frac{(2r-c)(r+c)^3}{2c(c+x)} - \frac{(2r+c)(r-c)^3}{2c(c-x)}\right]\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \quad \star \quad + \frac{2}{3}c\frac{c\cos\phi - x\cos\gamma}{PQ} - \frac{(2r-c)(r+c)^2}{6r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega')$$

$$- \frac{(2r+c)(r-c)^2}{6r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega')$$

$$= \quad \star \quad + \quad \star \quad - \frac{2}{3}r\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega) - \left(c - \frac{c^3}{3r^2}\right)\frac{d}{d\theta}(J_3' + \tfrac{1}{2}\Omega'), \quad (24)$$

the star replacing a term repeated.

Then, by addition,

$$\frac{1}{2G\rho}\frac{dX'}{d\theta} = -b\sin\phi\frac{dJ_1}{d\theta} - \frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2}\,.\,PQ\,.\,J_2\right) - \frac{d}{d\theta}(r - x\,.\,J_3)$$

$$-\tfrac{1}{3}a\cos\phi\,\frac{a}{PQ} - \tfrac{1}{3}(A\cos\phi - 2b\sin\phi)\,\frac{a\cos\theta}{PQ}$$

$$+\tfrac{1}{2}b\frac{d\Omega}{d\theta} + \tfrac{1}{3}r\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega) + \frac{c^3}{3r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega'). \quad (25)$$

So long as $\phi > \gamma$, $X' = X$, and J_2, J_3 vanish at both limits, as well as J_1; and

$$\frac{X}{G\rho} = -\tfrac{2}{3}a\cos\phi\int_0^\pi \frac{a\,d\theta}{PQ} + \tfrac{2}{3}(A\cos\phi - 2b\sin\phi)\int\frac{-a\cos\theta\,d\theta}{PQ}$$

$$+\Omega b\cos\phi + \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}$$

$$= -\frac{W}{G\sigma}\cos\phi + \tfrac{2}{3}Pa\cos\phi - \tfrac{2}{3}Q(A\cos\phi + b\sin\phi)$$

$$+\tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}. \quad (26)$$

5. But when $\phi < \gamma$, CP cuts the base AB, and the attraction X'' must be added of the spherical segment cut off by the plane BB' perpendicular to CP; and X'' is obtained by replacing J_2 and J_3 by π in (4), § 4, and integrating from x_2 to c; then

$$\frac{1}{G\rho}\frac{dX''}{dx} = 2\pi\left(1 - \frac{PL}{PQ}\right), \tag{1}$$

$$\begin{aligned}\frac{X''}{G\rho} &= 2\pi\int_{x_2}^{c}\left[1 - \frac{r-x}{(r^2+c^2-2rx)^{\frac{1}{2}}}\right]dx \\ &= 2\pi(c-x_2) + 2\pi\frac{(2r+c)(r-c)^2}{3r^2} + 2\pi\frac{rx_2 - 2r^2 + c^2}{3r^2}r_2.\end{aligned} \tag{2}$$

At the upper limit we must now take $J_3 = \pi$, $J_3' = -\pi$, and then

$$\frac{X}{G\rho} = \frac{X' + X''}{G\rho} = \text{same value as in (26), § 4,} \tag{3}$$

so that there is no discontinuity in X as P crosses the line CB.

As a verification for X, consider the case where P is at K or O, when $\phi = 0$:

$$\frac{X}{G\rho} = -\frac{2\pi a^3}{PA} + \Omega b + \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}. \tag{4}$$

At K, we must take

$$\left.\begin{array}{l} r = c, \quad b = -c(1-\cos\gamma), \quad \Omega = -2\pi(1-\sin\tfrac{1}{2}\gamma), \\ \Omega' = 4\pi + \Omega, \qquad \Omega + \Omega' = 4\pi\sin\tfrac{1}{2}\gamma, \end{array}\right\} \tag{5}$$

$$\frac{X}{G\rho} = 4\pi c\sin^2\tfrac{1}{2}\gamma - \tfrac{8}{3}\pi c\sin^3\tfrac{1}{2}\gamma = 2\pi\,.\,KO\left(1 - \tfrac{2}{3}\frac{KO}{KB}\right), \tag{6}$$

agreeing with the result of a direct integration, as for X'' in (2).

At O, we take

$$r = x_2 = c\cos\gamma, \quad b = 0, \quad \Omega = 2\pi, \quad \Omega' = -2\pi(1-\sin\gamma), \tag{7}$$

$$\frac{X}{G\rho} = -\tfrac{2}{3}\pi a + \tfrac{2}{3}\pi r - \tfrac{2}{3}\frac{\pi c^2}{r^2}(c-a) = \tfrac{2}{3}\pi c\left(-\sin\gamma + \cos\gamma - \frac{1}{1+\sin\gamma}\right), \tag{8}$$

agreeing, when the sign is changed, with the value obtained by integration.

For a hemisphere, $\gamma = \frac{1}{2}\pi$,

$$\frac{X}{G\rho} = 2\pi c(1 - \tfrac{1}{3}\sqrt{2}), \text{ at } K, \quad = -\pi c, \text{ at } O. \tag{9}$$

In the plane of the base of the hemisphere, $\phi = \frac{1}{2}\pi$,

$$\frac{X}{G\rho} = \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega' \frac{c^3}{r^2}, \tag{10}$$

$$r > c, \quad \Omega = 0, \quad \Omega' = 2\pi, \quad \frac{X}{G\rho} = \frac{2\pi c^3}{3r^2}, \tag{11}$$

$$r < c, \quad \Omega = 2\pi, \quad \Omega' = 0, \quad \frac{X}{G\rho} = \tfrac{2}{3}\pi r, \tag{12}$$

a verification.

6. If V denotes the gravitation potential of the segment or flat lens,

$$\frac{1}{G\rho}\frac{dV}{dr} = -\frac{X}{G\rho} = \frac{W}{G\sigma}\cos\phi - \tfrac{2}{3}Pa\cos\phi$$
$$+ \tfrac{2}{3}Q(A\cos\phi + b\sin\phi) - \tfrac{1}{3}\Omega r - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}, \tag{1}$$

$$\frac{1}{G\rho}\frac{dV}{r\,d\phi} = -\frac{Y}{G\rho} = -\frac{W}{G\sigma}\sin\phi + \tfrac{2}{3}Pa\sin\phi - \tfrac{1}{3}Q\frac{A^2 + a^2 + b^2}{A}\sin\phi, \tag{2}$$

$$\frac{1}{G\rho}\frac{dV}{dA} = -\frac{X}{G\rho}\sin\phi - \frac{Y}{G\rho}\cos\phi$$
$$= -\tfrac{1}{3}\Omega r\sin\phi - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}\sin\phi + \tfrac{1}{3}Q(b + b')$$
$$= -\tfrac{1}{3}(\Omega A - Qb) - \tfrac{1}{3}\frac{c}{r}(\Omega' A' - Q' b') \tag{3}$$

(an accent referring to the point P'),

$$\frac{1}{G\rho}\frac{dV}{db} = \frac{X}{G\rho}\cos\phi - \frac{Y}{G\rho}\sin\phi$$
$$= \frac{W}{G\sigma} - \tfrac{1}{3}\Omega r\cos\phi - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}\cos\phi + \tfrac{1}{3}Q(A + A') - \tfrac{2}{3}Pa$$
$$= \frac{W}{G\sigma} - \tfrac{1}{3}(\Omega r\cos\phi - QA) - \tfrac{1}{3}\frac{c}{r}(\Omega' r'\cos\phi - Q'A') - \tfrac{2}{3}Pa. \tag{4}$$

We shall find in the sequel, § 8, that these components are derivable from a potential V, where

$$\frac{V}{G\rho} = -\tfrac{1}{2}\frac{W}{G\sigma}b + \Omega(\tfrac{1}{2}c^2 - \tfrac{1}{6}r^2) + \tfrac{1}{3}\Omega'\frac{c^3}{r}$$
$$- \tfrac{1}{3}PaA\cot\phi + \tfrac{1}{3}QAc\cos\gamma. \tag{5}$$

The cyclic constant 4π in Ω and Ω' will give rise to the terms

$$4\pi\left(\tfrac{1}{2}c^2 - \tfrac{1}{6}r^2\right), \text{ and } \tfrac{4}{3}\pi\frac{c^3}{r}, \tag{6}$$

in $\dfrac{V}{G\rho}$, corresponding to the potential, inside and outside, of a solid sphere.

The Attraction of a Spherical Bowl.

7. The bowl is supposed constituted of a superficial density σ over the spherical surface of the segment; and a similar dissection by planes perpendicular to CP will show that X_1, Y_1, the components of attraction, will be given by

$\dfrac{1}{G\sigma}\dfrac{dY_1}{dx}$ = component attraction perpendicular to CP of the arc QRQ' in Fig. 2, allowing for the slanting section,

$$= \frac{2c\,.\,QN}{PQ^3}, \tag{1}$$

$$\frac{1}{G\sigma}\frac{dY_1}{d\theta} = 2ac\sin\phi\sin\theta\,\frac{a\sin\theta}{PQ^3}$$

$$= \frac{2ac}{A}\sin\phi\,\frac{d}{d\theta}\left(\frac{a\sin\theta}{PQ}\right) - 2\frac{c}{A}\sin\phi\,\frac{a\cos\theta}{PQ}, \tag{2}$$

$$\frac{Y_1}{G\sigma} = 2\frac{c}{A}\sin\phi\int_0^\pi \frac{-a\cos\theta\, d\theta}{PQ} = Q\frac{c}{r}, \tag{3}$$

$$\frac{1}{G\sigma}\frac{dX_1'}{dx} = \frac{2c\,.\,PL\,.\,J_2}{PQ^3} = \frac{2c(r-x)J_2}{(r^2+c^2-2rx)^{\frac{3}{2}}}$$

$$= -\frac{2c}{r^2}\frac{d}{dx}\left(\frac{c^2-rx}{PQ}\,.\,J_2\right) + \frac{2c}{r^2}\,.\,\frac{c^2-rx}{PQ}\,\frac{dJ_2}{dx}, \tag{4}$$

$$\frac{1}{G\sigma}\frac{dX_1'}{d\theta} = -\frac{2c}{r^2}\frac{d}{d\theta}\left(\frac{c^2-rx}{PQ}\,.\,J_2\right) + \frac{c^2}{r^2}\frac{d}{d\theta}\left(\Omega' + 2J_3'\right). \tag{5}$$

Then, if $\phi > \gamma$, $X_1' = X_1$, and J_2, J_3' are both zero at B, so that

$$\frac{X_1}{G\sigma} = \Omega'\frac{c^2}{r^2}. \tag{6}$$

But if $\phi < \gamma$, X_1'' must be added to X_1', where X'' is obtained by putting $J_2 = \pi$ in (4),

$$\frac{1}{G\sigma}\frac{dX_1''}{dx} = \frac{2\pi c\,.\,PL}{PQ^3}, \tag{7}$$

$$\frac{X_1''}{G\sigma} = 2\pi c\int_{x_2}^{c} \frac{r-x}{(r^2+c^2-2rx)^{\frac{3}{2}}}\,dx = \frac{2\pi c}{r^2}\left(\frac{c^2-rx_2}{PB}+c\right), \tag{8}$$

and with $J_3 = -\pi$ at B,

$$\frac{X_1}{G\sigma} = \frac{X_1' + X_1''}{G\sigma} = \Omega' \frac{c^2}{r^2}, \tag{9}$$

as before in (6), so that there is no discontinuity in X_1 as P crosses CB.

At the vertex K, where $\phi = 0$, $A = 0$,

$$PQ^2 = A^2 + a^2 + b^2 + 2Aa\cos\theta = KA^2\left(1 + \frac{2Aa}{KA^2}\cos\theta\right), \tag{10}$$

$$\frac{Y_1}{G\sigma} = \frac{2ac\sin\phi}{A\,.\,KA}\int_0^\pi \left(-\cos\theta + \frac{Aa\cos^2\theta}{KA^2}\ldots\ldots\right)d\theta = \frac{\pi a^2 c}{KA^3}\sin\phi, \tag{11}$$

so that the small oscillation of a smooth particle on the surface of the bowl near K is given by

$$\frac{d^2\phi}{dt^2} + n^2\sin\phi = 0, \text{ where } n^2 = \frac{Y}{c\sin\phi} = G\sigma\frac{\pi a^2}{KA^3}, \tag{12}$$

and the number of oscillations for second is $n \div 2\pi$.

The Potential of the Bowl and the Spherical Segment, a Flat Lens.

8. With the same dissection, denoting the potential of the bowl by U,

$$\frac{1}{G\sigma}\frac{dU'}{dx} = \text{potential at } P \text{ of the arc } QRQ' \times \sec CRL$$

$$= \frac{2c\,.\,J_2}{PQ} = -\frac{2c}{r}\frac{d}{dx}(PQ\,.\,J_2) + \frac{2c}{r}PQ\frac{dJ_2}{dx}, \tag{1}$$

$$\frac{1}{G\sigma}\frac{dU'}{d\theta} = -\frac{2c}{r}\frac{d}{d\theta}(PQ\,.\,J_2) + \frac{2c^2}{r}\,.\,\frac{c\cos\phi - x\cos\gamma}{c^2 - x^2}\,.\,PQ$$

$$= \quad \star \quad + \frac{2c^2}{r}\left[\frac{(r+c)^2}{2c(c+x)} + \frac{(r-c)^2}{2c(c-x)}\right]\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \quad \star \quad + c\frac{r+c}{r}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega')$$

$$+ c\frac{r-c}{r}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega')$$

$$= \quad \star \quad + c\frac{d}{d\theta}(2J_3 + \Omega) + \frac{c^2}{r}\frac{d}{d\theta}(2J_3' + \Omega'). \tag{2}$$

With $\phi > \gamma$, $U = U'$, and J_2, J_3, J_3' are zero at B;

$$\frac{U}{G\sigma} = \Omega c + \Omega'\frac{c^2}{r}. \tag{3}$$

But with $\phi < \gamma$, U'' must be added to U' to obtain U, where, replacing J_2 by π in (1),

$$\frac{1}{G\sigma}\frac{dU''}{dr} = \frac{2\pi c}{PQ}, \quad \frac{U''}{G\sigma} = 2\pi c\int_{x_2}^{c}\frac{dx}{PQ} = \frac{2\pi c}{r}(r_2 - r + c); \tag{4}$$

and now J_3 becomes π at B, and J_3' becomes $-\pi$; and then

$$\frac{U}{G\sigma} = \Omega c + \Omega' \frac{c^3}{r}, \tag{5}$$

so that there is no discontinuity in U as P crosses CB.

Expressed by its mass $M = 2\pi\sigma c \,.\, OK$,

$$\frac{U}{GM} = \frac{1}{2\pi \,.\, OK}\left(\Omega + \Omega' \frac{c}{r}\right). \tag{6}$$

We can now determine the potential V of the segment in a simple manner by considering it as a homogeneous function in the second degree of the dimensions, and by calculating the variation when the figure swells by variation of r and c, γ and ϕ remaining constant; and then

$$2V = r\frac{dV}{dr} + c\frac{dV}{dc} = -Xr + c\frac{dV}{dc}. \tag{7}$$

Now the variation dV, due to the change dc in c, is equivalent to the addition of the potential of a spherical bowl on the surface, of thickness dc and superficial density $\rho\, dc$, less the potential of a circular disc of radius $a = c\sin\gamma$, thickness $dc\cos\gamma$, and superficial density $\rho\, dc\cos\gamma$; so that, from (6), § 3, and (3), § 8,

$$\frac{1}{G\rho}\frac{dV}{dc} = \Omega c + \Omega' \frac{c^2}{r} - \frac{W}{G\sigma}\cos\gamma, \tag{8}$$

$$\frac{2V}{G\rho} = \frac{W}{G\sigma} r\cos\phi - \tfrac{2}{3} Par\cos\phi + \tfrac{2}{3} Qr(A\cos\phi + b\sin\phi)$$
$$- \tfrac{1}{3}\Omega r^2 - \tfrac{1}{3}\Omega' \frac{c^3}{r} + \Omega c^2 + \Omega' \frac{c^3}{r} - \frac{W}{G\sigma} c\cos\gamma, \tag{9}$$

which is equivalent to the statement in (5), § 6.

9. The potential V may be calculated directly, by the method employed for X and Y, from the relation

$$\frac{1}{G\rho}\frac{dV}{dx} = \text{potential of the circular segment } QRQ' \text{ at a point } P \text{ on its axis } CP. \tag{1}$$

The potential at P of the line element qq' is

$$\int_{-nq}^{nq}\frac{dz}{Pn}=\int\frac{dz}{\surd(Pn^2+z^2)}=\log\frac{Pq+nq}{Pq-nq}=2\,\text{th}^{-1}\frac{nq}{PQ}, \tag{2}$$

since $Pq=PQ$ round the periphery of the circle in Fig. 2.

The potential then at P of the segment QRQ' is

$$2\int_{LN}^{LR}\text{th}^{-1}\frac{\surd(LQ^2-y^2)}{PQ}dy=2y\,\text{th}^{-1}\frac{nq}{PQ}\Big]_{LN}^{LR}-2\int y\frac{-y\,dy\,.\,PQ}{(PQ^2-LQ^2+y^2)\surd(LQ^2-y^2)}$$

$$=-2LN\text{th}^{-1}\frac{NQ}{PQ}+2PQ\int\frac{dy}{\surd(LQ^2-y^2)}-2\int\frac{PQ\,.\,PL^2\,dy}{(PL^2+y^2)\surd(LQ^2-y^2)}$$

$$=-2LN\text{ch}^{-1}\frac{PQ}{PN}+2PQ\cos^{-1}\frac{LN}{LQ}-2\,PL\cos^{-1}\frac{PQ\,.\,LN}{PN\,.\,LQ}, \tag{3}$$

or, in the previous notation,

$$\frac{1}{G\rho}\frac{dV}{dx}=-\,2LN\,.\,J_1+2PQ\,.\,J_2-2\,PL\,.\,J_3. \tag{4}$$

Thence V was found by an integration by parts similar to those above; this was the method employed at first, but the work was very heavy, and it is omitted here, as the result obtained by the short method in (7), § 8, was found to be in agreement.

10. The well-known expression for the potential V of the homogeneous ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

may be interpreted in the same way, by treating V as a homogeneous function of the second degree in the linear scale of the ellipsoid, and in x, y, z, the coordinates of the point P; so that

$$2V=a\frac{dV}{da}+x\frac{dV}{dx}+y\frac{dV}{dy}+z\frac{dV}{dz}$$

$$=a\frac{dV}{da}-xX-yY-zZ. \tag{2}$$

Taking the components of attraction, X, Y, Z, as known,

$$\frac{X,\ Y,\ Z}{G\rho}=2xA_\lambda,\quad 2yB_\lambda,\quad 2zC_\lambda, \tag{3}$$

$$A_\lambda,\ B_\lambda,\ C_\lambda=\int_\lambda^\infty\frac{2\pi abc}{(\lambda+a^2,\ \lambda+b^2,\ \lambda+c^2)}\cdot\frac{d\lambda}{\surd P}, \tag{4}$$

$$P=4\,.\,\lambda+a^2\,.\,\lambda+b^2\,.\,\lambda+c^2,\quad A_\lambda+B_\lambda+C_\lambda=\frac{4\pi abc}{\surd P}; \tag{5}$$

the variation dV due to the change da may be considered the potential of a film in electrical equilibrium, of superficial density $\rho\frac{p}{a}da$, where p is the perpendicular from the centre on the tangent plane; and the theorem

$$a\frac{dV}{da}=\int_\lambda^\infty\frac{4\pi G\rho\,abc\,d\lambda}{\sqrt{P}}, \tag{6}$$

being taken as known, then

$$\frac{V}{G\rho}=\int_\lambda^\infty\left(1-\frac{x^2}{\lambda+a^2}-\frac{y^2}{\lambda+b^2}-\frac{z^2}{\lambda+c^2}\right)\frac{2\pi\,abc\,d\lambda}{\sqrt{P}} \tag{7}$$

is the result in consequence.

The Stokes Current Function.

11. In these investigations of the attraction of a body symmetrical about an axis Ox, the current function is useful, invented by Stokes (*Cambridge Phil. Soc. Trans.*, 1842). Denoting it by N, for a potential V, it satisfies the relations

$$\frac{dN}{dx}=2\pi y\frac{dV}{dy},\quad \frac{dN}{dy}=-2\pi y\frac{dV}{dx}, \tag{1}$$

$$\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=0, \tag{2}$$

while

$$\frac{d}{dx}\left(y\frac{dV}{dx}\right)+\frac{d}{dy}\left(y\frac{dV}{dy}\right)=0,\text{ or }-4\pi G\rho. \tag{3}$$

Thus in Maxwell's *E. and M.*, § 703, $M=2\pi QA$ is the current function of the magnetic potential Ω.

If the ellipsoid in (1), § 10, is of revolution about Ox, $b=c$,

$$\frac{V}{G\rho}=\int_\lambda^\infty\left(1-\frac{x^2}{\lambda+a^2}-\frac{y^2+z^2}{\lambda+b^2}\right)\frac{\pi ab^2\,d\lambda}{(\lambda+b^2)\sqrt{(\lambda+a^2)}}, \tag{4}$$

and we find

$$\begin{aligned}\frac{N}{2\pi G\rho}&=-xy^2B_\lambda-\tfrac{2}{3}\frac{\pi ab^2x^3}{(\lambda+a^2)^{\frac{3}{2}}}\\&=\tfrac{1}{2}xy^2A_\lambda-\frac{\pi ab^2xy^2}{(\lambda+b^2)\sqrt{(\lambda+a^2)}}-\tfrac{2}{3}\frac{\pi ab^2x^3}{(\lambda+a^2)^{\frac{3}{2}}},\end{aligned} \tag{5}$$

$$\frac{1}{2\pi G\rho}\frac{dN}{dx}=-2y^2B_\lambda=y\frac{dV}{dy},\quad \frac{1}{2\pi G\rho}\frac{dN}{dy}=2xyA_\lambda=-y\frac{dV}{dx}. \tag{6}$$

Thus the velocity function for motion of the ellipsoid along Ox with velocity U being

$$\phi = -\frac{Ux\,A_\lambda}{B_0 + C_0}, \quad \text{or } -\frac{Ux\,A_\lambda}{2\,B_0}, \quad \text{when } B_0 = C_0, \tag{7}$$

the current function is

$$\psi = -\frac{Uy^2\,B_\lambda}{2\,B_0}, \tag{8}$$

and when the ellipsoid is reduced to rest, and the fluid streams part,

$$\phi = -Ux\left(1 + \frac{A_\lambda}{2\,B_0}\right), \quad \psi = \tfrac{1}{2}\,Uy^2\left(1 - \frac{B_\lambda}{B_0}\right), \tag{9}$$

making $\psi = 0$ over the surface.

In the magnetic analogue, the spheroid, of soft iron of magnetic permeability μ, held with its axis parallel to a uniform magnetic field of strength X, will have

$$\phi = Xx\left[-1\,\frac{(\mu - 1)\,A_\lambda}{\mu\,A_0 + 2\,B_0}\right], \quad \psi = \tfrac{1}{2}\,Xy^2\left[1 + \frac{2\,(\mu - 1)\,B_\lambda}{\mu\,A_0 + 2\,B_0}\right] \tag{10}$$

in the exterior field, and in the interior

$$\phi = \frac{4\,\pi\,Xx}{\mu\,A_0 + 2\,B_0}, \quad \psi = \frac{2\,\pi\,\mu\,Xy^2}{\mu\,A_0 + 2\,B_0}; \tag{11}$$

reducing for $\mu = 0$ to the hydrodynamical case of (9).

For an oblate spheroid, putting

$$\lambda + a^2 = (b^2 - a^2)\cot^2\theta, \quad \lambda + b^2 = (b^2 - a^2)\csc^2\theta, \tag{12}$$

$$A_\lambda = \frac{2\,\pi\,a\,b^2}{(b^2 - a^2)^{\frac{3}{2}}}(\tan\theta - \theta), \quad A_0 = \frac{2\,\pi\,b^2}{(b^2 - a^2)^{\frac{3}{2}}}\left[\sqrt{(b^2 - a^2)} - a\cos^{-1}\frac{a}{b}\right], \tag{13}$$

$$B_\lambda = \frac{\frac{1}{2}\,\pi\,a\,b^2}{(b^2 - a^2)^{\frac{3}{2}}}(2\theta - \sin 2\theta), \quad B_0 = \frac{\pi\,a}{(b^2 - a^2)^{\frac{3}{2}}}\left[b^2\cos^{-1}\frac{a}{b} - a\sqrt{(b^2 - a^2)}\right]. \tag{14}$$

If v denotes the potential when the spheroid is insulated and electrified by a charge Q to potential v_0, with $G\sigma_0$ the electric density at A,

$$\frac{v}{G\sigma_0} = \frac{1}{G\rho}\,\frac{dV}{da} = 4\,\pi\,b\,\theta, \quad \frac{Q}{G\sigma_0} = \int \frac{p}{a}\,dS = 4\,\pi\,b^2, \tag{15}$$

so that the capacity is $b/\cos^{-1}\dfrac{a}{b}$.

For a disc, with $a = 0$,

$$A_0 = 2\,\pi, \quad B_0 = 0, \quad \frac{B_0}{a} = \frac{\pi^2}{2\,b}, \quad \frac{a\,A_0}{2\,B_0} = \frac{2\,b}{\pi}, \tag{16}$$

$$\sin\theta = \frac{AB}{PA + PB}, \tag{17}$$

and the capacity is $b/\frac{1}{2}\,\pi$ (*E. and M.*, § 151).

12. If L denotes the current function corresponding to the potential W of the circular disc AB, it was found in *Trans. A. M. S.*, pp. 501, 513,

$$\frac{L}{2\pi G\sigma} = -\tfrac{1}{2} Pab - \tfrac{1}{2} QAb + \tfrac{1}{2}\Omega(A^2 - a^2), \tag{1}$$

giving

$$\frac{1}{2\pi G\sigma}\frac{dL}{db} = -QA = \frac{1}{G\sigma}\frac{dW}{dA}A,$$

$$\frac{1}{2\pi G\sigma}\frac{dL}{dA} = \Omega A = -\frac{1}{G\sigma}\frac{dW}{db}A,$$

$$\frac{1}{2\pi G\sigma}\frac{dL}{da} = -Pb - \Omega a, \tag{2}$$

and this last is the current function of P, the potential function of the circumference of the circle AB.

On reference to *Trans. A. M. S.*, p. 513, we notice that $\frac{L}{2\pi G\sigma p}$ will give the coefficient of mutual induction between the helix, employed in the Ampere Balance of Ayrton and Viriamu Jones, of pitch p, height b, and radius a, and a coaxial circle of radius A in the plane of one end of the helix (*Phil. Trans.*, 1891, 1907).

The current function M of the potential U of the bowl is found to be given by

$$\frac{M}{2\pi G\sigma} = -Pac + QAc + \Omega c^2\cos\gamma + \Omega' c^2\cos\phi$$

$$= -\frac{W}{G\sigma}c + \frac{U}{G\sigma}r\cos\phi, \tag{3}$$

satisfying the conditions

$$\left.\begin{aligned} \frac{1}{2\pi G\sigma}\frac{dM}{dr} &= Qc\sin\phi = A\frac{Y_1}{G\sigma} = -A\frac{dU}{rd\phi}, \\ \frac{1}{2\pi G\sigma}\frac{dM}{rd\phi} &= -\Omega'\frac{c^2}{r}\sin\phi = -A\frac{X_1}{G\sigma} = A\frac{dU}{dr}. \end{aligned}\right\} \tag{4}$$

If N denotes the current function of V, the potential of the solid segment, treated as a homogeneous function of the third degree, it can be determined, in the same manner as V, from

$$3N = r\frac{dN}{dr} + c\frac{dN}{dc} = 2\pi Yr^2\sin\phi + c\frac{dN}{dc}, \tag{5}$$

where

$$\frac{dN}{dc} = M - L\cos\gamma, \tag{6}$$

$$\frac{3N}{2\pi G\rho} = \left(\frac{W}{G\sigma} - S\right)A^2 - \frac{L}{2\pi G\sigma}c\cos\gamma + \frac{Mc}{2\pi G\sigma}; \tag{7}$$

and this value of N is found to verify in giving

$$\frac{1}{2\pi}\frac{dN}{db} = A\frac{dV}{dA}, \quad \frac{1}{2\pi}\frac{dN}{dA} = -A\frac{dV}{db}. \tag{8}$$

We can now write

$$\begin{aligned}\frac{V}{G\rho} &= -\tfrac{1}{3}\frac{L}{2\pi G\sigma} - \tfrac{1}{3}\frac{W}{G\sigma}(b + c\cos\gamma) + \tfrac{1}{3}\frac{Uc}{G\sigma} \\ &= \tfrac{1}{3}\frac{U - W\cos\gamma}{G\sigma}c - \tfrac{1}{3}\frac{L}{2\pi G\sigma} - \tfrac{1}{3}\frac{Wb}{G\sigma}.\end{aligned} \tag{9}$$

13. In these differentiations it is useful to note that

$$\left.\begin{aligned} a\frac{dP}{dA} - A\frac{dQ}{dA} - b\frac{d\Omega}{dA} = 0, \quad a\frac{dP}{db} - A\frac{dQ}{db} - b\frac{d\Omega}{db} = 0, \\ a\frac{dP}{da} - A\frac{dQ}{da} - b\frac{d\Omega}{da} = 0, \end{aligned}\right\} \tag{1}$$

making

$$\frac{1}{G\sigma}\frac{dW}{dA} = -Q, \quad \frac{1}{G\sigma}\frac{dW}{db} = -\Omega, \quad \text{in } \frac{W}{G\sigma} = Pa - QA - \Omega b; \tag{2}$$

and other useful differentiations are:

$$\left.\begin{aligned} \frac{dP}{dA} = 2\int_0^\pi \frac{-a(A + a\cos\theta)\,d\theta}{PQ^3}, \quad \frac{dP}{db} = 2\int\frac{-ab\,d\theta}{PQ^3}, \\ \frac{dP}{da} = 2\int\frac{(Aa\cos\theta + A^2 + b^2)\,d\theta}{PQ^3}, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \frac{dQ}{dA} = 2\int\frac{a\cos\theta\,(A + a\cos\theta)\,d\theta}{PQ^3}, \quad \frac{dQ}{db} = 2\int\frac{ab\cos\theta\,d\theta}{PQ^3}, \\ \frac{dQ}{da} = 2\int\frac{-(Aa\cos\theta + A^2 + b^2)\cos\theta\,d\theta}{PQ^3}, \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \frac{d\Omega}{dA} = 2\int\frac{ab\cos\theta\,d\theta}{PQ^3} = \frac{dQ}{db}, \\ \frac{d\Omega}{db} = 2\int\frac{-Aa\cos\theta - a^2}{PQ^3}\,d\theta = -\frac{dQ}{dA} - \frac{Q}{A}, \end{aligned}\right\} \tag{5}$$

$$a\frac{d\Omega}{da} = -A\frac{d\Omega}{dA} - b\frac{d\Omega}{db} = 2\int\frac{a^2b\,d\theta}{PQ^3}, \quad \frac{d\Omega}{da} = -\frac{dP}{db}. \tag{6}$$

$$a\frac{dP}{da} = -A\frac{dP}{dA} - b\frac{dP}{db}, \quad a\frac{dQ}{da} = -A\frac{dQ}{dA} - b\frac{dQ}{db}. \tag{7}$$

Denoting

$$PA^2 . PB^2 = (A^2 + a^2 + b^2)^2 - 4A^2a^2 \text{ by } D, \tag{8}$$

we find that, expressed by P and Q,

$$\frac{d\Omega}{dA} = \frac{dQ}{db} = -Pa\frac{2Ab}{D} + Qb\frac{A^2+a^2+b^2}{D}, \tag{9}$$

$$\frac{d\Omega}{db} = -\frac{dQ}{dA} - \frac{Q}{A} = Pa\frac{A^2-a^2-b^2}{D} - QA\frac{A^2-a^2+b^2}{D}, \tag{10}$$

$$\frac{dP}{dA} = -PA\frac{A^2-a^2+b^2}{D} + Qa\frac{A^2-a^2-b^2}{D}, \tag{11}$$

$$\frac{dP}{db} = -Pb\frac{A^2+a^2+b^2}{D} + QA\frac{2ab}{D}. \tag{12}$$

Also

$$SA^2 = \tfrac{2}{3}PaA^2 - \tfrac{1}{3}QA(A^2+a^2+b^2), \quad S = -\tfrac{1}{3}\frac{D}{Ab}\frac{d\Omega}{dA}, \tag{13}$$

derivable by integration of

$$\frac{d}{d\theta}(Aa\sin\theta . PQ) = -3\frac{A^2a^2\sin\theta}{PQ} + 2\frac{A^2a^2}{PQ} + (A^2+a^2+b^2)\frac{Aa\cos\theta}{PQ}, \tag{14}$$

$$\frac{dSA^2}{db} = -QAb, \quad \frac{dSA^2}{dA} = (Pa - QA)A = \frac{WA}{G\sigma} + \Omega Ab. \tag{15}$$

Again, since

$$\frac{1}{G\sigma}\frac{dU}{dr} = \frac{d\Omega}{dr}c + \frac{d\Omega'}{dr}.\frac{c^2}{r} - \Omega'\frac{c^2}{r^2} = -\frac{X_1}{G\sigma} = -\Omega'\frac{c^2}{r^2}, \tag{16}$$

$$\frac{1}{G\sigma}\frac{dU}{rd\phi} = \frac{d\Omega}{rd\phi}c + \frac{d\Omega'}{rd\phi}\frac{c^2}{r} = -\frac{Y_1}{G\sigma} = -Q\frac{c}{r}, \tag{17}$$

it follows that

$$r\frac{d\Omega}{dr} + c\frac{d\Omega'}{dr} = 0, \qquad r\frac{d\Omega}{d\phi} + c\frac{d\Omega'}{d\phi} = -Qr, \tag{18}$$

$$r\frac{d\Omega}{dA} + c\frac{d\Omega'}{dA} = -Q\cos\phi, \quad r\frac{d\Omega}{db} + c\frac{d\Omega'}{db} = -Q\sin\phi. \tag{19}$$

This collection of formulas is useful for reference in the differentiation of the potential and current function.

14. These expressions for V and N give the potential and current function of a flat lens, and addition or subtraction will give them for a lens, concavo-convex in Fig. 5, or double convex as in Fig. 6.

Then if C_1 is the centre and c_1 the radius of the second spherical surface,

$$\frac{V - V_1}{G\rho} = \tfrac{1}{3}\frac{W}{G\sigma}\, CC_1 + \tfrac{1}{3}\frac{Uc - U_1 c_1}{G\sigma}, \tag{1}$$

for the concave lens of Fig. 5; and in Fig. 6 for the convex lens

$$\frac{V + V_1}{G\rho} = \tfrac{1}{3}\frac{W}{G\sigma}\, CC_1 + \tfrac{1}{3}\frac{Uc + U_1 c_1}{G\sigma}. \tag{2}$$

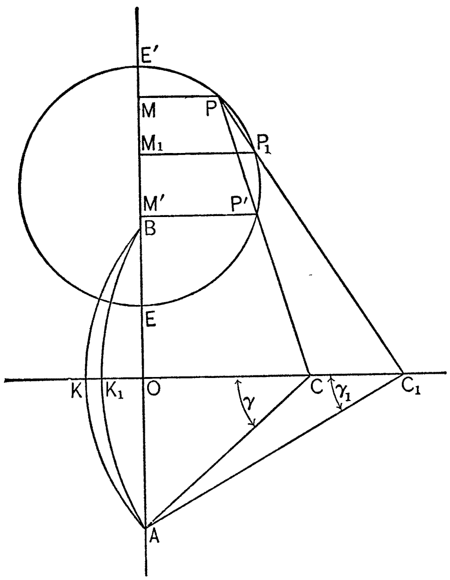

FIG. 5.

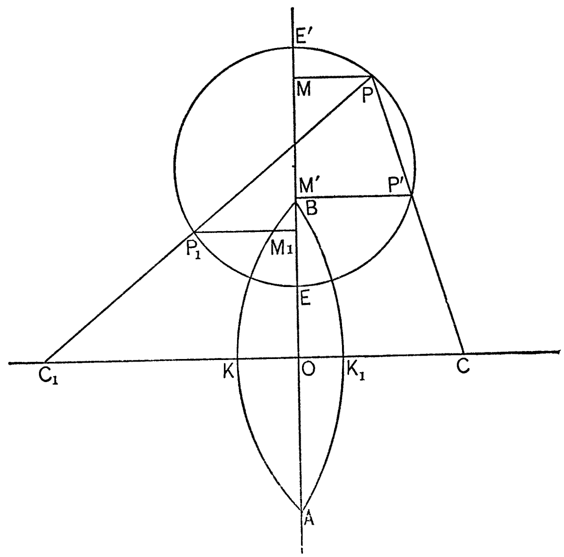

FIG. 6.

For example, with a complete sphere, where $CC_1 = 0$, $c = c_1 = a$,

$$\frac{V + V_1}{G\rho} = \tfrac{1}{3}\frac{U + U_1}{G\sigma}\, a, \tag{3}$$

and the cyclic constants of Ω and Ω' must be so adjusted as to make, in the exterior space

$$\frac{U + U_1}{G\sigma} = 4\pi\frac{a^2}{r}, \quad \frac{V + V_1}{G\rho} = \tfrac{4}{3}\pi\frac{a^3}{r}. \tag{4}$$

But inside the sphere

$$\frac{U + U_1}{G\sigma} = 4\pi a, \quad \frac{V + V_1}{G\rho} = 4\pi\left(\tfrac{1}{2}a^2 - \tfrac{1}{6}r^2\right), \tag{5}$$

requiring a careful adjustment of the cyclic constant; herein is the difficulty of expressing the single-valued potential by means of the multiple-valued Ω and Ω'.

Similarly for the current function of the concave lens of Fig. 5,

$$\frac{N - N_1}{2\pi G\rho} = \tfrac{1}{3}\frac{L}{2\pi G\sigma} CC_1 + \tfrac{1}{3}\frac{Mc - M_1 c_1}{2\pi G\sigma}, \tag{6}$$

as is evident from (7), § 12.

15. For a thin concave lens,

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \tfrac{1}{3}\frac{W}{G\sigma}\frac{a}{\sin^2\gamma} + \tfrac{1}{3}\frac{1}{G\sigma}\frac{dUc}{d\gamma}, \tag{1}$$

$$\frac{1}{2\pi G\rho}\frac{dN}{d\gamma} = \tfrac{1}{3}\frac{L}{2\pi G\sigma}\frac{a}{\sin^2\gamma} + \tfrac{1}{3}\frac{1}{2\pi G\sigma}\frac{dMc}{d\gamma}. \tag{2}$$

Keeping A, b, Ω constant, and varying c, r, ϕ, b', A', Ω' with γ, we find

$$\left.\begin{aligned} &\frac{dc}{d\gamma} = \frac{-a\cos\gamma}{\sin^2\gamma}, \quad \frac{dr}{d\gamma} = \frac{-a\cos\phi}{\sin^2\gamma}, \quad \frac{r\,d\phi}{d\gamma} = \frac{a\sin\phi}{\sin^2\gamma}, \\ &\frac{db'}{d\gamma} = \frac{A'^2 - a^2 - b'^2}{a}, \quad \frac{dA'}{d\gamma} = -\frac{2A'b'}{a}, \end{aligned}\right\} \tag{3}$$

$$\begin{aligned} \frac{d\Omega'}{d\gamma} &= \frac{d\Omega'}{db'}\frac{db'}{d\gamma} + \frac{d\Omega'}{dA'}\frac{dA'}{d\gamma} \\ &= \left(P'a\frac{A'^2 - a^2 - b'^2}{D'} - Q'A'\frac{A'^2 - a^2 + b'^2}{D'}\right)\frac{A'^2 - a^2 - b'^2}{a} \\ &\quad + \left(-P'a\frac{2A'b'}{D'} + Q'b'\frac{A'^2 + a^2 + b'^2}{D'}\right)\frac{-2A'b'}{a} \\ &= \frac{P'a - Q'A'}{a}, \end{aligned} \tag{4}$$

$$\frac{1}{G\sigma}\frac{dU}{d\gamma} = -\frac{U}{G\sigma}\frac{\cos\gamma}{\sin\gamma} + \frac{W'}{G\sigma}\frac{a}{r\sin^2\gamma}, \tag{5}$$

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \tfrac{1}{3}\frac{W}{G\sigma}\frac{a}{\sin^2\gamma} - \tfrac{1}{3}\frac{U}{G\sigma}\frac{2a\cos\gamma}{\sin^2\gamma} + \tfrac{1}{3}\frac{W'}{G\sigma}\frac{ac}{r\sin^2\gamma}. \tag{6}$$

If m denotes the mass of the flat lens,

$$m = \tfrac{1}{3}\rho\pi a^3\frac{(1-\cos\gamma)^2(2+\cos\gamma)}{\sin^3\gamma}, \quad \frac{dm}{d\gamma} = \frac{\rho\pi a^3}{(1+\cos\gamma)^2}, \tag{7}$$

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \frac{1}{G}\frac{dV}{dm}\frac{\pi a^3}{(1+\cos\gamma)^2}. \tag{8}$$

Similarly, from (3), § 12,

$$\frac{1}{2\pi G\sigma}\frac{dM}{d\gamma}=-\frac{M\cos\gamma+M'\cos\phi}{2\pi G\sigma\sin\gamma}-\frac{U}{G\sigma}\frac{a\sin^2\phi}{\sin^2\gamma}$$

$$=-\frac{M}{2\pi G\sigma}\frac{\cos\gamma}{\sin\gamma}+\frac{W'\cos\phi-U}{G\sigma}\frac{a}{\sin^2\gamma}, \tag{9}$$

$$\frac{1}{2\pi G\rho}\frac{dN}{d\gamma}=\tfrac{1}{3}\frac{L}{2\pi G\sigma}\frac{a}{\sin^2\gamma}-\tfrac{1}{3}\frac{M}{2\pi G\sigma}\frac{2a\cos\gamma}{\sin^2\gamma}+\tfrac{1}{3}\frac{W'\cos\phi-U}{G\sigma}\frac{ac}{\sin^2\gamma}; \tag{10}$$

and comparing this (10) with (6), we notice that $W'\cos\phi-U$ should be the current function of $\dfrac{W'}{r}$, which proves to be the case, by a differentiation in verification.

16. When $\gamma=0$, $c, r=\infty$, $\dfrac{c}{r}=1$, $W=W'=U$, $L=M$, and $\dfrac{dV}{d\gamma}$, $\dfrac{dN}{d\gamma}$ take an indeterminate form.

To evaluate them for the lens, which is now a flat disc, in which the thickness and superficial density at a point Q varies as $Q'A\,.\,Q'B=a^2-y^2$ at a radius $OQ'=y$, and for mass μ the superficial density $\sigma=\dfrac{2\mu}{\pi a^4}(a^2-y^2)$, we calculate the potential v directly by the dissection into ring elements; and then

$$\frac{v}{G\mu}=\frac{1}{G}\frac{dV}{dm}=\frac{2}{\pi a^4}\int_{\theta=0}^{2\pi}\int_{y=0}^{a}(a^2-y^2)\frac{y\,dy\,d\theta}{PQ'}, \tag{1}$$

and, with $\gamma=0$,

$$\frac{1}{G\rho}\frac{dV}{d\gamma}=\frac{1}{G}\frac{dV}{dm}\cdot\frac{\pi a^3}{4}. \tag{2}$$

Integrating with respect to y, with

$$PQ'^2=y^2+2Ay\cos\theta+A^2+b^2, \tag{3}$$

$$\begin{aligned}\int_0^a(-y^3+a^2y)\frac{dy}{PQ'}=&-\tfrac{1}{3}PQ^3+\tfrac{1}{3}PO^3+\tfrac{3}{2}Aa\cos\theta\,.\,PQ\\&+(-\tfrac{5}{2}A^2\cos^2\theta+A^2+a^2+b^2)(PQ-PO)\\&+A\cos\theta\,(A^2-a^2-\tfrac{3}{2}b^2-\tfrac{5}{2}A^2\sin^2\theta)\,I,\end{aligned} \tag{4}$$

where

$$I=\int_0^a\frac{dy}{PQ'}=\mathrm{ch}^{-1}\frac{PQ}{PZ}-\mathrm{ch}^{-1}\frac{PO}{PZ}=\mathrm{sh}^{-1}\frac{a+A\cos\theta}{PZ}-\mathrm{sh}^{-1}\frac{A\cos\theta}{PZ}, \tag{5}$$

and PZ is the perpendicular on QO,

$$PZ^2 = A^2 \sin^2\theta + b^2, \qquad PO^2 = A^2 + b^2, \tag{6}$$

$$\frac{dI}{d\theta} = -\frac{Aa\cos\theta + A^2 + b^2}{PZ^2} \cdot \frac{A\sin\theta}{PQ} + \frac{PO \, . \, A\sin\theta}{PZ^2}. \tag{7}$$

Writing

$$\begin{aligned} A\cos\theta\,(A^2 - a^2 - \tfrac{3}{2}b^2 - \tfrac{5}{2}A^2\sin^2\theta)\,I \\ = \frac{d}{d\theta}\left[A\sin\theta\,(A^2 - a^2 - \tfrac{3}{2}b^2 - \tfrac{5}{6}A^2\sin^2\theta)\,I\right] \\ - A\sin\theta\,(A^2 - a^2 - \tfrac{3}{2}b^2 - \tfrac{5}{6}A^2\sin^2\theta)\frac{dI}{d\theta}, \end{aligned} \tag{8}$$

and integrating again with respect to θ, we find after considerable reduction of a previous character that, with $\gamma = 0$,

$$\frac{2a}{G\rho}\frac{dV}{d\gamma} = \tfrac{2}{3}\frac{L}{2\pi G\sigma}b - \tfrac{2}{3}\frac{W}{G\sigma}(A^2 - a^2 - b^2) + \tfrac{2}{3}SA^2; \tag{9}$$

and W can also be calculated in this manner directly, from a dissection of the circle AB into concentric rings.

Thence we infer

$$\frac{2a}{2\pi G\rho}\frac{dN}{d\gamma} = -\frac{L}{2\pi G\sigma} \cdot \tfrac{1}{2}(A^2 - a^2) - \frac{W}{G\sigma}A^2 b + \tfrac{1}{2}SA^2 b. \tag{10}$$

For differentiating, with

$$\frac{1}{2\pi G\sigma}\frac{dL}{db} = \frac{A}{G\sigma}\frac{dW}{dA} = -QA, \quad \frac{1}{2\pi G\sigma}\frac{dL}{dA} = -\frac{A}{G\sigma}\frac{dV}{db} = \Omega A, \tag{11}$$

we find

$$\frac{2a}{2\pi G\rho}\frac{d^2N}{db\,d\gamma} = -2\frac{W}{G\sigma}A^2 + \tfrac{2}{3}SA^2 = \frac{2aA}{G\rho}\frac{d^2V}{dA\,d\gamma}, \tag{12}$$

$$\frac{2a}{2\pi G\rho}\frac{d^2N}{dA\,d\gamma} = -\frac{2LA}{2\pi G\sigma} - 2\frac{WAb}{G\sigma} = -\frac{2aA}{G\rho}\frac{d^2V}{db\,d\gamma}. \tag{13}$$

17. Going back again to the concave lens of finite thickness, we find

$$\begin{aligned} \frac{1}{2\pi G\rho}\frac{d(N - N_1)}{db} &= \tfrac{1}{3}A\frac{dW}{dA} \cdot CC_1 + \tfrac{1}{3}A\left(c\frac{dU}{dA} - c_1\frac{dU_1}{dA}\right) \\ &= \tfrac{1}{3}QA(b' - b_1) - \tfrac{1}{3}\frac{\Omega' c^3\sin^3\phi - \Omega_1 c_1^3\sin^3\phi_1}{A} \\ &= A\frac{d(V - V_1)}{dA}, \end{aligned} \tag{1}$$

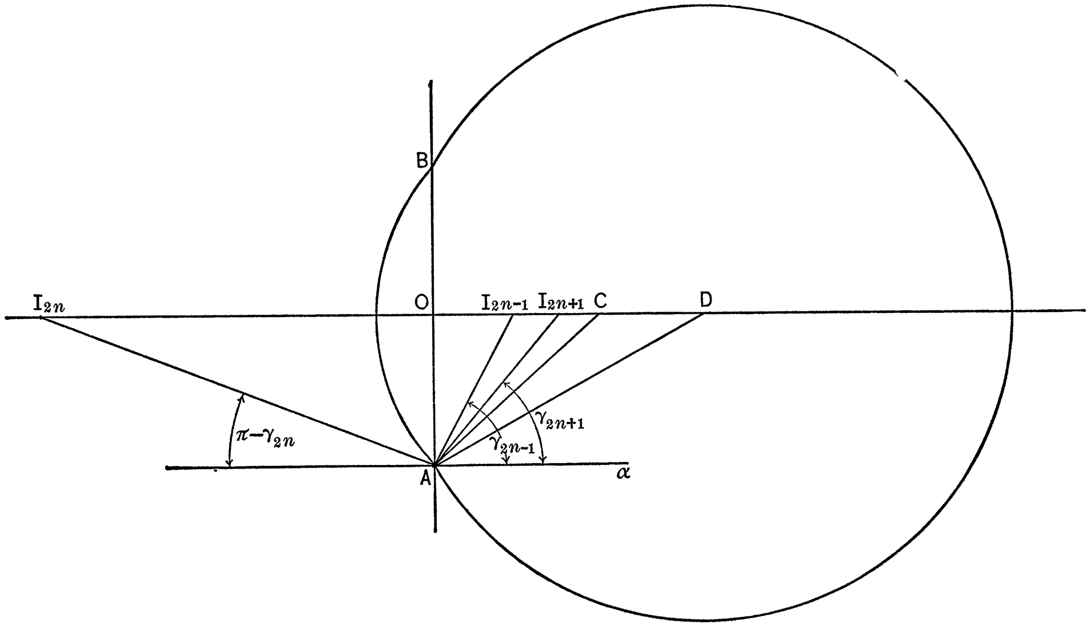

FIG. 7.

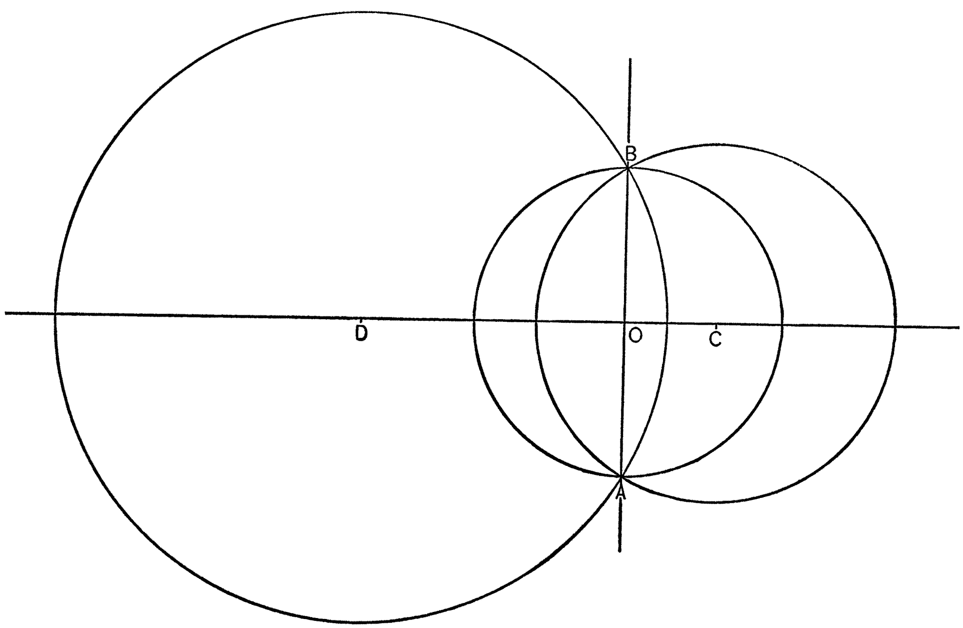

FIG. 8.

$$\frac{1}{2\pi G\rho}\frac{d(N-N_1)}{dA} = -\tfrac{1}{3}A\frac{dW}{db}\cdot CC_1 - \tfrac{1}{3}A\left(c\frac{dU}{db} - c_1\frac{dU_1}{db}\right)$$

$$= \tfrac{1}{3}QA(A'-A_1) - \tfrac{1}{3}A\left(\Omega r + \Omega'\frac{c^3}{r^2}\right)\cos\phi$$

$$+ \tfrac{1}{3}A\left(\Omega r_1 + \Omega_1\frac{c_1^3}{r_1^2}\right)\cos\phi_1$$

$$= -A\frac{d(V-V_1)}{db}, \tag{2}$$

giving the components of attraction of the concave lens.

The difficulty is now to return to the flat lens, where $c_1 = \infty$, introducing the indeterminate form $0 \times \infty$; thus the expression for the potential and current function of the flat lens in (7) and (9), § 12, requires

$$\mathrm{lt}\,(U_1 - W\cos\gamma_1)\,c_1 = \frac{L}{2\pi} + Wb, \quad \mathrm{lt}\,\frac{L\cos\gamma_1 - M_1}{2\pi}c_1 = \left(\frac{W}{G\sigma} - S\right)A^2; \tag{3}$$

which are not obvious, and require careful treatment.

18. From the expression in (6), § 15, it should not be impossible now to obtain the potential of the thin curved lens from an integration of its ring-shaped elements, and so obtain a direct method which employs the dissection into elements symmetrical about the axis OC.

The expressions in (1), (2), § 17, will give the magnetic potential of the lens, and its current function of vector potential, when magnetized uniformly; and by analogy with the ellipsoid in § 11, the same expressions should help towards the determination of the velocity function when the lens moves through liquid, like the bob of a pendulum cutting the air; but some modification is required still, before the requisite conditions are satisfied.

Take two spheres in Fig. 7, centre C and D, intersecting in the circle AB at an angle $\gamma - \delta = 2\alpha$, γ and δ denoting the angle OCA and ODA; and put $\gamma + \delta = 2\beta$, the bisector of the angle DAC making an angle β with OC.

Next take the succession of points $I_1, I_2, \ldots, I_n, \ldots$ on CD which are the electric images or inverse points in succession with respect to the two spherical surfaces, denoted by the centre C and D; beginning with I_1 at C, and I_2 the image or inverse point of I_1 in the sphere D, I_3 the image of I_2 in the sphere C, and so on; and denote the angle OI_nA or I_nNa by γ_n.

Then since

$$D I_{2n-1} \,.\, D I_{2n} = D A^2, \quad \text{and} \quad C I_{2n} \,.\, C I_{2n+1} = C A^2, \tag{1}$$

$$O I_{2n} A = C A I_{2n-1} = D A I_{2n+1}, \tag{2}$$

$$\pi - \gamma_{2n} = \gamma_{2n-1} - \delta = \gamma_{2n+1} - \gamma, \tag{3}$$

$$\left.\begin{aligned} \gamma_{2n+1} - \gamma_{2n-1} &= \gamma - \delta = 2\alpha, \\ \gamma_{2n+1} - \gamma_1 &= 2n\alpha, \end{aligned}\right\} \tag{4}$$

and with $\gamma_1 = \gamma = \alpha + \beta$,

$$\left.\begin{aligned} \gamma_{2n+1} &= (2n+1)\alpha + \beta, \\ \gamma_{2n} &= \pi - 2n\alpha. \end{aligned}\right\} \tag{5}$$

If f_n denotes the distance of I_n to the right of O in Fig. 7, and C_n denotes the distance CI_n, c_1 being c, the radius of the sphere C,

$$f_{2n} = a \cot(\pi - 2n\alpha) = -a \cot 2n\alpha, \quad f_{2n+1} = a \cot[(2n+1)\alpha + \beta], \tag{6}$$

$$c_{2n} = a \csc 2n\alpha, \quad c_{2n+1} = a \csc[(2n+1)\alpha + \beta]. \tag{7}$$

If $J_1, J_2, \ldots, J_n, \ldots$ denotes the series of images, beginning with J_1 at D, then by a change of sign of α, caused by the interchange of γ and δ,

$$g_{2n} = a \cot 2n\alpha, \quad g_{2n+1} = -a \cot[(2n+1)\alpha - \beta], \tag{8}$$

$$d_{2n} = a \csc 2n\alpha, \quad d_{2n+1} = a \csc[(2n+1)\alpha - \beta], \tag{9}$$

with f and c changed into g and d.

If the spheres cut at an angle $2\alpha = \frac{\pi}{m}$, or $\frac{r\pi}{m}$, an aliquot part of π, the number of images is finite and $2m+1$, analogous to the finite number of images seen by reflexion in two plane mirrors at an angle $\pi r/m$.

Thus if m is an even number $2p$, $f_p = g_p = 0$, $f_{p+r} = g_{p-r}$; and if m is odd, $2p+1$,

$$f_{2p+1} = g_{2p-1}, \quad f_{2p+r} = g_{2p-r}.$$

(R. A. Herman, "A Problem in Fluid Motion," *Quarterly Journal of Mathematics*, Vol. XXII, 1887.)

Denoting the distance PI_n, PJ_n by r_n, s_n, consider first the potential function

$$U = \frac{c_1}{r_1} - \frac{c_2}{r_2} + \frac{c_3}{r_3} \cdots + \frac{d_3}{s_3} - \frac{d_2}{s_2} + \frac{d_1}{s_1}, \tag{10}$$

due to the series of a finite number of images I_n and J_n, beginning with I_1 at C and J_1 at D, and joining up in the middle.

Over the sphere C,

$$r_1 = c_1, \quad \frac{c_2}{r_2} = \frac{c_3}{r_3}, \quad \ldots, \quad \frac{d_2}{r_2} = \frac{d_1}{r_1}; \tag{11}$$

and over the sphere D,

$$s_1 = d_1, \quad \frac{d_2}{s_2} = \frac{d_3}{s_3}, \quad \ldots, \quad \frac{c_2}{r_2} = \frac{c_1}{r_1}; \tag{12}$$

so that over the surface of the sphere, $U = 1$, and U will serve for the potential in exterior space of an electrical distribution over the outside surface of the intersecting spheres.

Change the circular functions into hyperbolic, to obtain the electrification of two non-intersecting spheres, with images along the line of centres AB, A and B being the limiting points of the two spheres, and then Maxwell's coefficients of induction and capacity (*E. and M.*, § 173) are given by

$$q_{aa} = \sum_{n=1}^{\infty} c_{2n-1}, \qquad q_{ab} = -\Sigma\, c_{2n} = -\Sigma\, d_{2n}, \qquad q_{bb} = \Sigma\, d_{2n-1}; \tag{13}$$

the images being now infinite in number, but condensed ultimately at A and B.

Next consider the potential function V and its current function N, given by

$$\frac{V}{G\rho} = \frac{c_1^3}{r_1} - \frac{c_2^3}{r_2} - \ldots - \frac{d_2^3}{s_2} + \frac{d_1^3}{s_1} = \Sigma\,(-1)^{n-1}\left(\frac{c_n^3}{r_n} + \frac{d_n^3}{s_n}\right), \tag{14}$$

$$\frac{N}{2\pi\, G\rho} = c_1^3 \cos PI_1O - c_2^3 \cos PI_2O - \ldots - d_2^3 \cos PJ_2O + d_1^3 \cos PJ_1O$$

$$= \Sigma\,(-1)^{n-1}\left(c_n^3\,\frac{f_n - x}{r_n} + d_n^3\,\frac{g_n - x}{s_n}\right), \tag{15}$$

due to a series of spheres, of density $\pm\frac{3}{4}\rho$, centres at I_n and J_n, all intersecting in the same circle AB, like a series of lenses; a sphere being condensed afterwards into a particle at its centre.

Then with these particles replaced by magnetic molecules,

$$\frac{1}{2\pi\, G\rho}\,\frac{dN}{dx} = -y^2\,\Sigma\,(-1)^{n-1}\left(\frac{c_n^3}{r_n^3} + \frac{d_n^3}{s_n^3}\right), \tag{16}$$

reducing, as in (11) and (12), to $-y^2$ over the sphere C and D; so that for a velocity u of the two spheres in the direction OC through infinite liquid, the current function ψ and velocity function ϕ will be given by

$$\frac{\psi}{\frac{1}{2}u} = \frac{1}{2\pi\, G\rho}\,\frac{dN}{dx}, \qquad \frac{\phi}{\frac{1}{2}u} = \frac{1}{G\rho}\,\frac{dV}{dx}. \tag{17}$$

The kinetic energy T of the exterior liquid, taken of density σ, will be found by integrating over the spherical surfaces; and will be given by

$$\frac{T}{\frac{1}{2}u^2} \text{ (the effective inertia)} = \tfrac{1}{2}\sigma \int \frac{\Phi}{\frac{1}{2}u} \cos PCO\, dS$$

$$= \tfrac{1}{2}\sigma \int \frac{1}{G\rho}\frac{dV}{dx}\, 2\pi y\, dy = -\tfrac{1}{2}\sigma \int \frac{1}{G\rho}\frac{dN}{dy}\, dy$$

$$= -\tfrac{1}{2}\sigma \left[\frac{N}{G\rho}\right] + \int \frac{\sigma}{G\rho}\frac{dN}{dx}\, dx = -\tfrac{1}{2}\sigma \left[\frac{N}{G\rho}\right] - \sigma \int \pi y^2 dx; \qquad (18)$$

and in passing from the left at H to the right at K of the spheres, N increases by $2\,G\rho\pi\,(c_1^3 - c_2^3 - \ldots\ldots - d_2^3 + d_1^3)$; so that

$$\text{the effective inertia} = \sigma\pi\,(c_1^3 - c_2^3 - \ldots\ldots - d_2^3 + d_1^3)$$
$$- \text{ the displacement of liquid.} \qquad (19)$$

Thus when the spheres coalesce, $m = 1$, $c_1 = d_1 = c$, and

$$\text{the effective inertia} = 2\sigma\pi c^3 - \tfrac{4}{3}\sigma\pi c^3 = \tfrac{2}{3}\sigma\pi c^3$$
$$= \text{half the displacement of liquid;} \qquad (20)$$

a verification.

For two orthogonal spheres, as in Fig. 8, $m = 2$, and there is only one image I_2 or J_2 at O, $c_2 = d_2 = a$; and V is the potential of three lenses, an interior double-convex lens, and two exterior concavo-convex, with two interspaces. This is analogous to the case of the squarable meniscus of Antiphon and Hippocrates (Simplicius, F. Rudio, Teubner, 1907).

19. In the reduction in (16), § 2, of

$$\Omega = \int_0^{2\pi} \frac{QN^2}{PN^2} \cdot \frac{b\,d\theta}{PQ} \qquad (1)$$

to Legendre's standard form, and its numerical expression by his Table IX, the consideration of imaginary parameters is avoided by the substitution

$$\left.\begin{aligned} \frac{PN^2}{QN^2} &= \frac{s-\sigma}{\sigma - s_3}, \qquad \frac{PQ^2}{QN^2} = \frac{s - s_3}{\sigma - s_3}, \\ \frac{QN^2}{PQ^2} &= \frac{\sigma - s_3}{s - s_3} = \frac{a^2 \sin^2\theta}{2Aa\cos\theta + A^2 + a^2 + b^2}. \end{aligned}\right\} \qquad (2)$$

This is equivalent to a quadric transformation of Mr. Hill's reduction; and the quadratic for $\cos\theta$ is

$$\left.\begin{aligned}
&\cos^2\theta + 2\frac{A}{a}\frac{\sigma - s_3}{s - s_3}\cos\theta + \frac{A^2 + a^2 + b^2}{a^2}\frac{\sigma - s_3}{s - s_3} - 1 = 0,\\
&\left(\cos\theta + \frac{A}{a}\frac{\sigma - s}{s - s_3}\right)^2 = 1 - \frac{A^2 + a^2 + b^2}{a^2}\frac{\sigma - s_3}{s - s_3} + \frac{A^2}{a^2}\left(\frac{\sigma - s_3}{s - s_3}\right)^2\\
&\qquad = \frac{s - s_1 \,.\, s - s_2}{(s - s_3)^2},
\end{aligned}\right\} \quad (3)$$

with

$$\left.\begin{aligned}
\frac{r_1^2 + r_2^2}{2a^2} &= \frac{A^2 + a^2 + b^2}{a^2} = \frac{s_1 - s_3}{\sigma - s_3} + \frac{s_2 - s_3}{\sigma - s_3},\\
\frac{r_1^2 - r_2^2}{2a^2} &= 2\frac{A}{a} = 2\frac{\sqrt{(s_1 - s_3 \,.\, s_2 - s_3)}}{\sigma - s_3},\\
\frac{r_1,\ r_2}{a} &= \frac{\sqrt{(s_1 - s_3)} \pm \sqrt{(s_2 - s_3)}}{\sqrt{(\sigma - s_3)}},\\
\frac{r_1 + r_2}{2a} &= \sqrt{\frac{s_1 - s_3}{\sigma - s_3}}, \qquad \frac{r_1 - r_2}{2a} = \sqrt{\frac{s_2 - s_3}{\sigma - s_3}},\\
\frac{b}{a} &= \frac{\sqrt{(s_1 - \sigma \,.\, \sigma - s_2)}}{\sigma - s_3}.
\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}
\cos\theta &= \frac{-\sqrt{(s_1 - s_3 \,.\, s_2 - s_3)} + \sqrt{(s - s_1 \,.\, s - s_2)}}{s - s_3},\\
\sin\theta &= \frac{\sqrt{(s_2 - s_3 \,.\, s - s_1)} + \sqrt{(s_1 - s_3 \,.\, s - s_2)}}{s - s_3},\\
\frac{PQ}{a} &= \sin\theta\sqrt{\frac{s - s_3}{\sigma - s_3}} = \frac{\sqrt{(s_2 - s_3 \,.\, s - s_1)} + \sqrt{(s_1 - s_3 \,.\, s - s_2)}}{\sqrt{(s - s_3 \,.\, \sigma - s_3)}},\\
-\sin\theta\frac{d\theta}{ds} &= \frac{[\sqrt{(s_2 - s_3 \,.\, s - s_1)} + \sqrt{(s_1 - s_3 \,.\, s - s_2)}]^2}{2(s - s_3)^2\sqrt{(s - s_1 \,.\, s - s_2)}},\\
-\frac{d\theta}{ds} &= \frac{\sqrt{(s_2 - s_3 \,.\, s - s_1)} + \sqrt{(s_1 - s_3 \,.\, s - s_2)}}{2(s - s_3)\sqrt{(s - s_1 \,.\, s - s_2)}},\\
\frac{a\,d\theta}{PQ} &= -\frac{\sqrt{(\sigma - s_3)}\,ds}{\sqrt{S}}.
\end{aligned}\right\} \quad (5)$$

As θ increases from 0 to π, s diminishes from ∞ to s_1 and increases again to ∞, so that, as in *Trans. A. M. S.*, p. 491,

$$\Omega = 4\int_{s_1}^{\infty}\frac{\sqrt{(s_1 - \sigma \,.\, \sigma - s_2 \,.\, \sigma - s_3)}}{s - \sigma}\frac{ds}{\sqrt{S}} = 2\pi f - 4K\,\mathrm{zn}\,(1 - f)K', \quad (6)$$

the equivalent of Legendre's formula (m'), *Fonctions elliptiques*, p. 138, with $\frac{1}{4}\Omega$ the equivalent of the left-hand side of Legendre's equation, when we put Legendre's

$$\sin^2\phi = \frac{s_1 - s_3}{s - s_3}, \text{ and his } n = -\Delta^2(\theta, k'),\ n\sin^2\phi = -\frac{\sigma - s_3}{s - s_3}, \tag{7}$$

$$\left.\begin{aligned} &\sin^2\theta = \operatorname{sn}^2(1-f)K' = \frac{s_1 - \sigma}{s_1 - s_2}, \quad \operatorname{cn}^2(1-f)K' = \frac{\sigma - s_2}{s_1 - s_2}, \\ &\operatorname{dn}^2(1-f)K' = \frac{\sigma - s_3}{s_1 - s_3} = \frac{4a^2}{(r_1 + r_2)^2}, \quad \varkappa = \sqrt{\frac{s_2 - s_3}{s_1 - s_3}} = \frac{r_1 - r_2}{r_1 + r_2}; \end{aligned}\right\} \tag{8}$$

and then

$$\operatorname{dn} fK' = \frac{r_1 - r_2}{2a}, \quad \operatorname{sn}^2 fK' = \frac{1 - \operatorname{dn}^2 fK'}{\varkappa'^2} = \frac{(r_1 + r_2)^2 - 4A^2}{4r_1 r_2} = \sin^2\chi, \tag{9}$$

denoting the angle PEE' by χ, so that $\chi = \operatorname{am} fK'$, and then if PB cuts the circle EE' again in p, $\operatorname{am}(1-f)K' = E'Ep$. For

$$\frac{AE}{BE} = \frac{AE'}{BE'} = \frac{r_1}{r_2}, \quad \frac{OE'}{OA} = \frac{r_1 + r_2}{r_1 - r_2} = \frac{1}{\varkappa}, \quad \frac{OE}{OA} = \frac{r_1 - r_2}{r_1 + r_2} = \varkappa, \tag{10}$$

$$\frac{EE'}{a} = \frac{1}{\varkappa} - \varkappa = \frac{4r_1 r_2}{r_1^2 - r_2^2} = \frac{r_1 r_2}{Aa}, \tag{11}$$

$$\left.\begin{aligned} &E'M = OE' - OM = a\frac{r_1 + r_2}{r_1 - r_2} - A = \frac{(r_1 + r_2)^2}{4A} - A, \\ &\sin^2\chi = \frac{E'M}{EE'} = \frac{(r_1 + r_2)^2 - 4A^2}{4r_1 r_2}. \end{aligned}\right\} \tag{12}$$

As P moves round the circle $E'PE$, $\varkappa$ and K do not change, but f increases from 0 to 1, and Ω from 0 at E' to 2π at E.

Expressed in terms of $E'\chi$ and $F'\chi$ of Legendre's Table IX, the accent denoting the comodulus $\varkappa'$,

$$fK' = F'\chi, \qquad \operatorname{zn} fK' = E'\chi - fH', \tag{13}$$

$$\operatorname{zn}(1-f)K' = \varkappa'^2 \frac{\sin\chi\cos\chi}{\Delta'\chi} - \operatorname{zn} fK', \tag{14}$$

so that, employing Legendre's relation,

$$\tfrac{1}{2}\pi = KH' + K'H - KK', \tag{15}$$

$$\tfrac{1}{4}\Omega = H \,.\, F'\chi - K(F'\chi - E'\chi) - K\varkappa'^2 \frac{\sin\chi\cos\chi}{\Delta'\chi}; \tag{16}$$

or, more simply, denoting the angle $E'Ep$ by θ,

$$\tfrac{1}{4}\Omega = \tfrac{1}{2}\pi + (K - H)F'\theta - KE'\theta, \tag{17}$$

as in Legendre's equation (m'), p. 138.

Denoting the angle $P'EE'$ by $\chi' = \mathrm{am} f' K'$,

$$\Omega' = 2\pi f' - 4K \mathrm{zn}(1 - f')K'; \tag{18}$$

and Ω, Ω' have a cyclic constant 2π for a circuit of the circle $E'PE$.

But as P and P' describe the circle in opposite direction, the cyclic constants cancel in the expression of V; otherwise, if a cyclic constant could exist in a gravity potential, work could be obtained from a circuit, or in the popular expression, perpetual motion would be possible.

The potential and attraction is a single-valued function, but expressed by Ω and Ω', functions having a cyclic constant 4π for a circuit of the circle EE'; and care must be taken to adjust the cyclic constants at any point P, although they cancel when P has made a circuit of the circle EE' in one direction, and P' in the opposite.

Thus as P moves in the direction PE, and Ω increases from zero at E', Ω may be taken to have gained 4π in passing through E', and 4π must be deducted, so that Ω should be zero again at E' at the end of the circuit.

So, too, P' moves in the opposite direction, starting from zero at E'; and Ω' must lose 4π, and have 4π added in passing through E.

Compare the gain or loss of a day in going round the world, moving east or west.

20. According to this reduction in § 19 of the elliptic integral

$$P = \int_0^\pi \frac{2a\,d\theta}{PQ} = 4\sqrt{\frac{\sigma - s_3}{s_1 - s_3}} \int_{s_1}^\infty \frac{\sqrt{(s_1 - s_3)}\,ds}{\sqrt{S}}$$

$$= 4K \mathrm{dn}(1 - f)K' = \frac{8Ka}{r_1 + r_2}; \tag{1}$$

and with

$$\left.\begin{aligned} s - s_3 &= (s_1 - s_3)\frac{1}{\mathrm{sn}^2 u}, \quad & s - s_2 &= (s_1 - s_3)\frac{\mathrm{dn}^2 u}{\mathrm{sn}^2 u}, \\ s - s_1 &= (s_1 - s_3)\frac{\mathrm{cn}^2 u}{\mathrm{dn}^2 u}, \quad & \frac{\sqrt{(s_1 - s_3)}\,ds}{\sqrt{S}} &= du, \end{aligned}\right\} \tag{2}$$

$$\cos\theta = -\varkappa\,\mathrm{sn}^2 u + \mathrm{cn}\,u\,\mathrm{dn}\,u, \quad \sin\theta = (\varkappa\,\mathrm{cn}\,u + \mathrm{dn}\,u)\,\mathrm{sn}\,u; \tag{3}$$

and u grows from 0 to $2K$ as θ increases from 0 to π, so that

$$Q = \int_0^\pi \frac{-2a\cos\theta\,d\theta}{PQ} = 2\sqrt{\frac{\sigma - s_3}{s_1 - s_3}} \int_0^{2K} (\varkappa\,\mathrm{sn}^2 u - \mathrm{cn}\,u\,\mathrm{dn}\,u)\,du$$

$$= 4\frac{\mathrm{dn}(1 - f)K'}{\varkappa} \int_0^K (1 - \mathrm{dn}^2 u)\,du$$

$$= \frac{4}{\mathrm{dn} fK'}(K - H) = \frac{8a}{r_1 - r_2}(K - H) = 2\frac{r_1 + r_2}{A}(K - H), \tag{4}$$

H denoting the complete elliptic integral of the second kind, to modulus $\varkappa$; and then

$$\int_0^\pi \frac{2QN^2}{PQ}d\theta = 2a\int \sin^2\theta \frac{a\,d\theta}{PQ}$$

$$= 2a \operatorname{dn}(1-f)K' \int_0^{2K} (\varkappa \operatorname{cn} u + \operatorname{dn} u)^2 \operatorname{sn}^2 u\,du$$

$$= \frac{8a^2}{r_1+r_2}\int_0^K (\varkappa^2 \operatorname{cn}^2 u + \operatorname{dn}^2 u)\operatorname{sn}^2 u\,du$$

$$= \frac{8a^2}{r_1+r_2}\cdot\frac{(1+\varkappa^2)H-(1-\varkappa^2)K}{3\varkappa^2}. \tag{5}$$

21. In Maxwell's notation, *E. and M.*, § 701,

$$M = 2\pi QA = 4\pi(K-H)(r_1+r_2), \qquad \varkappa = \frac{r_1-r_2}{r_1+r_2}, \tag{1}$$

his second expression when corrected; but if a return is made to the modulus c employed in Maxwell's first transformation, and in Mr. Hill's reduction, a preparation is made of the expression of Ω by means of the theorems in § 47, (1), (4), (9), p. 505, *Trans. A. M. S.*, 1907, where with

$$I_4 = \sin^{-1}\frac{QN.PM}{PN.MQ} = \cos^{-1}\frac{MN.PQ}{PN.MQ}$$

$$= \text{complement of the angle between the planes } PQN,\ PQM, \tag{2}$$

a differentiation gives

$$\frac{dI_4}{d\theta} = \frac{QN^2}{PM^2}\cdot\frac{b}{PQ} + \frac{Aa\cos\theta+a^2}{MQ^2}\cdot\frac{b}{PQ}$$

$$= \tfrac{1}{2}\frac{d\Omega}{d\theta} + \frac{\tfrac{1}{2}b}{PQ} - \frac{\tfrac{1}{2}(A^2-a^2)}{MQ^2}\cdot\frac{b}{PQ}, \tag{3}$$

which exhibits the addition of the imaginary parameters in Mr. Hill's treatment.

The reduction to a standard form is made, without any rearrangement of the constituents, by the substitution (*Trans. A. M. S.*, § 26, p. 477):

$$\left.\begin{aligned}
PQ^2 = r^2 &= A^2 + 2Aa\cos\theta + a^2 + b^2 = m^2(t_1 - t),\\
PA^2 = r_1^2 &= (A+a)^2 + b^2 = m^2(t_1 - t_3),\\
PB^2 = r_2^2 &= (A-a)^2 + b^2 = m^2(t_1 - t_2),\\
r_1^2 - r^2 &= 2Aa(1-\cos\theta) = m^2(t - t_3),\\
r^2 - r_2^2 &= 2Aa(1+\cos\theta) = m^2(t_2 - t),\\
2Aa\sin\theta &= m^2\sqrt{(t_2 - t\,.\,t - t_3)}, \quad 2Aa\sin\theta\,d\theta = m^2\,dt,\\
d\theta &= \frac{dt}{\sqrt{(t_2-t\,.\,t-t_3)}}, \quad \frac{d\theta}{PQ} = \frac{dt}{m\sqrt{(t_1-t\,.\,t_2-t\,.\,t-t_3)}} = \frac{2\,dt}{m\sqrt{T}},
\end{aligned}\right\} \tag{4}$$

r now denoting PQ, and m a homogeneity factor; and then

$$P = \int_0^\pi \frac{2\,a\,d\theta}{PQ} = \frac{4\,a}{m}\int_{t_3}^{t_2}\frac{dt}{\sqrt{T}} = \frac{4\,a}{r_1}\int\frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}} = \frac{4\,F a}{r_1}, \tag{5}$$

where F denotes the elliptic quarter period to the modulus c (not to be confused with the radius of the spherical surface), where

$$c^2 = \frac{t_2 - t_3}{t_1 - t_3} = \frac{4\,A\,a}{r_1^2}, \qquad c' = \frac{r_2}{r_1}; \tag{6}$$

and

$$M = 2\pi\,QA = 2\pi\int_0^\pi \frac{-\,2\,A\,a\cos\theta\,d\theta}{PQ} = 2\pi m\int_{t_3}^{t_2}(2\,t - t_2 - t_3)\frac{dt}{\sqrt{T}}$$

$$= 2\pi m\int[t_1 - t_3 + t_1 - t_2 - 2\,(t_1 - t)]\frac{dt}{\sqrt{T}}. \tag{7}$$

We now put

$$t_1 - t = (t_1 - t_3)\,\mathrm{dn}^2 u, \quad t_2 - t = (t_2 - t_3)\,\mathrm{cn}^2 u, \quad t - t_3 = (t_2 - t_3)\,\mathrm{sn}^2 u, \tag{8}$$

so that

$$\theta = 2\,\omega = 2\,\mathrm{am}\,(u,\,c), \quad \omega = ABQ, \text{ on Fig. 3}, \tag{9}$$

$$M = 2\pi r_1\int_0^K (1 + c'^2 - 2\,\mathrm{dn}^2 u)\,du$$

$$= 2\pi r_1\,[(2 - c^2)\,F - 2\,E], \tag{10}$$

agreeing with Maxwell's first expression for M in *E. and M.*, § 701, when his sign is changed; E denotes here the complete second elliptic integral to the modulus c.

Other useful formulas are:

$$\int_0^{2\pi} PQ\,d\theta = 4\,r_1\int_0^{\frac{1}{2}\pi}\Delta\omega\,d\omega = 4\,E\,r_1, \tag{11}$$

$$P = \frac{4\,F a}{r_1} = \frac{8\,K a}{r_1 + r_2}, \quad F = (1 + \varkappa)\,K, \quad K = \tfrac{1}{2}\,(1 + c')\,F, \tag{12}$$

$$P\,\frac{b}{a} = 4\,K\varkappa'^2\,\frac{\mathrm{sn}\,f K'\,\mathrm{cn}\,f K'}{\mathrm{dn}\,f K'} = 4\,F c'\,\mathrm{sn}\,2\,f F'. \tag{13}$$

22. When P is close to the circle AB, at B, r_2 is small and P is large; writing it

$$P=\int_0^{2\pi}\frac{a\,d\theta}{PQ}=\int_0^{\frac{1}{2}\pi}\frac{4a\,d\omega}{PQ}=\int\frac{4a\sin\omega\,d\omega}{PQ}+\int\frac{4a(1-\sin\omega)\,d\omega}{PQ}, \quad (1)$$

the first integral

$$\int_\omega^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{PQ}=\frac{4a}{\surd(r_1^2-r_2^2)}\operatorname{ch}^{-1}\frac{PQ}{r_2},\quad \int_0^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{PQ}=2\sqrt{\frac{a}{A}}\operatorname{ch}^{-1}\frac{r_1}{r_2}$$

$$=2\sqrt{\frac{a}{A}}\log\frac{r_1+2\surd(Aa)}{r_2},\quad \text{ultimately } 2\log\frac{4a}{r_2}, \quad (2)$$

with $r_1=2a$, $A=a$; and

$$\int_0^{\frac{1}{2}\pi}\frac{4a(1-\sin\omega)\,d\omega}{PQ}<\frac{4a}{r_1}\int\frac{1-\sin\omega}{\cos\omega}d\omega=2\int\frac{\cos\omega}{1+\sin\omega}d\omega \text{ or } 2\log 2, \quad (3)$$

so that, when r_2 is small, we may take

$$P=2\log\frac{8a}{r_2}+\text{small terms}, \quad (4)$$

$$P-Q=\int_0^{2\pi}\frac{a(1+\cos\theta)\,d\theta}{PQ}$$

$$=\frac{8a}{r_1}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\omega\,d\omega}{\Delta\omega}<\frac{8a}{r_1}\int\cos\omega\,d\omega \text{ or } \frac{8a}{r_1}, \quad (5)$$

and replacing r_1 by $2a$,

$$Q=2\log\frac{8a}{r_2}-4+\text{small terms}=2\log\frac{8a}{e^2r_2}+\text{small terms}, \quad (6)$$

$$M=2\pi QA=4\pi a\log\frac{8a}{e^2r_2}+\text{small terms}. \quad (7)$$

23. In this reduction of the third elliptic integral, we put

$$\left.\begin{array}{l} MQ^2=A^2+2Aa\cos\theta+a^2=m^2(\tau-t),\quad PM^2=b^2=m^2(t_1-\tau), \\ MA^2=(A+a)^2=m^2(\tau-t_2),\quad MB^2=(A-a)^2=m^2(\tau-t_3), \end{array}\right\} \quad (1)$$

and with

$$\operatorname{sn}^2 2fF'=\frac{t_1-\tau}{t_1-t_2}=\frac{b^2}{r_1^2}=\sin^2\psi,\quad \psi=\operatorname{am}(2fF',c'), \quad (2)$$

where ψ denotes the angle PBE', growing from 0 to π as f increases from 0 to 1,

$$\int_0^\pi\frac{\frac{1}{2}(A^2-a^2)}{MQ^2}\cdot\frac{bd\theta}{PQ}=\int_{t_3}^{t_2}\frac{\surd(t_1-\tau\,.\,\tau-t_2\,.\,\tau-t_3)}{\tau-t}\cdot\frac{dt}{\surd T}$$

$$=\pi f+F\operatorname{zn}2fF', \quad (3)$$

in accordance with Legendre's equation (m'); thus from (3), § 21,

$$\Omega = 2 I_4 + \int_{t_3}^{t_2} \frac{2\sqrt{(t_1 - \tau \,.\, \tau - t_2 \,.\, \tau - t_3)}}{\tau - t} \frac{dt}{\sqrt{T}} - \int \frac{2\sqrt{(t-\tau)}\,dt}{\sqrt{T}}$$

$$= 2 I_4 + 2\pi f + 2 F \operatorname{zn} 2 f F' - 2 F c' \operatorname{sn} 2 f F', \tag{4}$$

agreeing with (6), § 19, provided $I_4 = 0$, and

$$2 F c' \operatorname{sn} 2 f F' - 2 F \operatorname{zn} 2 f F' = 4 K \operatorname{zn} (1 - f) K', \tag{5}$$

$$2 F c' \operatorname{sn} 2 f F' + 2 F \operatorname{zn} 2 f F' = 4 K \operatorname{zn} f K', \tag{6}$$

a theorem of the quadric transformation of the zeta function.

The result of (2), § 12, will represent the current function or lines of magnetic force of the electric current circulating round the circumference AB, or of the circular plate AB magnetized normally; and this magnetic plate is equivalent to a compound plate composed of the superposition of two thin plates of superficial density $\pm \sigma$.

When these two plates are drawn apart on the same axis to a distance b, the result is equivalent to an integration with respect to b, and L is obtained for the pair of end-plates, or for the cylinder magnetized longitudinally; also for the equivalent solenoid made by a current sheet of electricity circulating circumferentially, giving the same magnetic field as the helical current of the Ampere balance.

In the hydrodynamical interpretation, the difference of the two values of L for an end of the solenoid will give the lines of flow of liquid, circulating through the solenoidal tube.

Then, with the former method of integration by parts,

$$\frac{L}{2\pi G \sigma} = \int_0^b - Q A\, db = \int_0^b \int_0^{2\pi} \frac{A a \cos\theta \, d\theta \, db}{PQ} = \int_0^{2\pi} A a \cos\theta \operatorname{sh}^{-1} \frac{b}{MQ} d\theta$$

$$= \left[A a \sin\theta \operatorname{sh}^{-1} \frac{b}{MQ} \right]_0^{2\pi} - \int \frac{A^2 a^2 \sin^2\theta}{MQ^2} \cdot \frac{b\, d\theta}{PQ}$$

$$= \int \left[\tfrac{1}{2} A a \cos\theta - \tfrac{1}{4} (A^2 + a^2) + \tfrac{1}{4} \frac{(A^2 - a^2)^2}{MQ^2} \right] \frac{b\, d\theta}{PQ}, \tag{7}$$

which agrees with (1), § 12, when the result of (3), § 23, is employed.

24. Denoting the integral in (3), § 23, by $\Pi(MQ)$, then by addition of (3), § 4, and (2), § 21,

$$J_4 = J_3 + I_4 = -\Pi(MQ) + \int \frac{\frac{1}{2} b\, d\theta}{PQ} + \int \frac{c(r-x)(c\cos\phi - x\cos\gamma)}{c^2 - x^2} \cdot \frac{d\theta}{PQ}, \quad (1)$$

$$J_4 = \sin^{-1}\frac{PQ \, . \, NQ}{LQ \, . \, MQ} \sin\phi = \cos^{-1}\frac{(2A\cos\phi + b\sin\phi)a\cos\theta + (A^2 + a^2)\cos\phi + Ab\sin\phi}{LQ \, . \, MQ}$$

$=$ the complement of the angle between the planes PQM, PQL. (2)

Similarly for the point P', and

$$\Pi(M'Q) = \int \frac{\frac{1}{2}(A'^2 - a^2)}{M'Q^2} \cdot \frac{b'\, d\theta}{PQ} = \pi f' + F \operatorname{zn} 2f' F', \quad (3)$$

$$J_4' = J_3' + I_4' = -\Pi(M'Q) + \int \frac{\frac{1}{2} b'\, d\theta}{PQ}$$
$$+ \int \frac{(c^2 - rx)(c\cos\phi - x\cos\gamma)}{c^2 - x^2} \cdot \frac{d\theta}{PQ}. \quad (4)$$

Thence, by addition and subtraction,

$$J_4 + J_4' + \Pi(MQ) + \Pi(M'Q)$$
$$= \tfrac{1}{2}(b + b') \int \frac{d\theta}{PQ} + \int \frac{(r+c)(c\cos\phi - x\cos\gamma)}{c+x} \cdot \frac{d\theta}{PQ}$$
$$= [\tfrac{1}{2}(b+b') - (r+c)\cos\gamma] \int \frac{d\theta}{PQ}$$
$$+ \int \frac{c(r+c)(\cos\phi + \cos\gamma)}{c+x} \cdot \frac{d\theta}{PQ}, \quad (5)$$

$$J_4 - J_4' + \Omega(MQ) - \Omega(M'Q)$$
$$= \tfrac{1}{2}(b - b') \int \frac{d\theta}{PQ} + \int \frac{(r-c)(c\cos\phi - x\cos\gamma)}{c-x} \cdot \frac{d\theta}{PQ}$$
$$= [\tfrac{1}{2}(b-b') + (r-c)\cos\gamma] \int \frac{d\theta}{PQ}$$
$$+ \int \frac{c(r-c)(\cos\phi - \cos\gamma)}{c-x} \cdot \frac{d\theta}{PQ}; \quad (6)$$

thus showing the addition and subtraction of the parameters of the third elliptic integrals, $\Omega(c)$ and $\Omega(-c)$, where

$$\left.\begin{aligned} \Omega(c) &= \int \frac{c(r-c)(\cos\gamma - \cos\phi)}{c-x} \frac{d\theta}{PQ}, \\ \Omega(-c) &= \int \frac{c(r+c)(\cos\gamma + \cos\phi)}{c+x} \frac{d\theta}{PQ}. \end{aligned}\right\} \quad (7)$$

Expressed by the variable $r = PQ$, or by $\Delta\omega$, with

$$R = r_1^2 - r^2 \,.\, r^2 - r_2^2, \qquad r = r_1 \Delta\omega, \qquad \omega = \tfrac{1}{2}\theta, \tag{8}$$

and reinstating Mr. Hill's x',

$$\Omega(c) = \int \frac{2cx'(x'-c)(\cos\gamma - \cos\phi)}{r^2 - (x'-c)^2} \cdot \frac{dr}{\sqrt{R}}$$

$$= \int \frac{2cx'(x'-c)(\cos\gamma - \cos\phi)}{r_1^2\Delta^2\omega - (x'-c)^2} \cdot \frac{1}{r_1}\frac{d\omega}{\Delta\omega}, \tag{9}$$

$$\Omega(-c) = \int \frac{2cx'(x'+c)(\cos\gamma + \cos\phi)}{(x'+c)^2 - r^2} \cdot \frac{dr}{\sqrt{R}}$$

$$= \int \frac{2cx'(x'+c)(\cos\gamma + \cos\phi)}{(x'+c)^2 - r_1^2\Delta^2\omega} \cdot \frac{1}{r_1}\frac{d\omega}{\Delta\omega}, \tag{10}$$

two elliptic integrals of the third kind, in Legendre's normal form, to a modulus c as in §§ 21, 23, not to be confused with the radius of the spherical surface.

The exploration of the field can be carried out for simple aliquot values of f, such as $f = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots$, when $\operatorname{zn} fK'$ can be expressed as an algebraical function, as shown in *Trans. A. M. S.*, § 60, p. 521.

Denoting the angle PAE' by ψ',

$$\sin\psi' = c'\sin\psi, \quad \cos\psi' = \operatorname{dn} 2fF', \quad \frac{\tan\psi'}{\tan\psi} = \frac{A-a}{A+a} = c'\operatorname{sn}(1-2f)F'; \tag{11}$$

$$\Omega = 2\pi f + 2F\operatorname{zn} 2fF' - 2F\sin\psi'. \tag{12}$$

The stereographic coordinates u and v are useful to employ, such that

$$b + Ai = a\tan\tfrac{1}{2}(u + vi), \qquad b, A = a\,\frac{\sin u, \operatorname{sh} v}{\operatorname{ch} v + \cos u}, \tag{13}$$

$$e^v = \frac{AP}{PB} = c', \qquad u = APB = \psi - \psi' = \pi - 2\theta, \qquad \psi + \psi' = 2\chi, \tag{14}$$

since PE bisects APB; and $a\csc u$ is the radius of the circle round ABP.

Thence the formulas of the quadric transformation,

$$\psi = \tfrac{1}{2}\pi - \theta + \chi, \qquad \sin\psi = \cos(\theta - \chi) = (1+\varkappa)\frac{\sin\chi\cos\chi}{\Delta\chi}, \tag{15}$$

$$\operatorname{sn} 2fF' = (1+\varkappa)\,\frac{\operatorname{sn} fK'\operatorname{cn} fK'}{\operatorname{dn} fK'}. \tag{16}$$

Thus when $f=\frac{1}{2}$, $A=a$, $\psi=\frac{1}{2}\pi$, $\sin\psi'=c'$, so that APB is the modular angle for c; and from (4), (5), (6), § 23, and (12), § 21,

$$4K\,\mathrm{zn}\,\tfrac{1}{2}K'=2Fc', \qquad \mathrm{zn}\,\tfrac{1}{2}K=\tfrac{1}{2}(1-\varkappa),$$

$$\Omega=\pi-2Fc'=\pi-2K(1-\varkappa),$$

$$\frac{L}{2\pi G\sigma}=\tfrac{1}{2}(P+Q)ab=ab\int_0^\pi\frac{a(1-\cos\theta)\,d\theta}{PQ}=\frac{4a^2b}{r_1}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\omega}{\Delta\omega}d\omega$$

$$=\frac{4a^2b}{r_1}\,\frac{F-E}{c^2}=(F-E)\,br_1=(F-E)\,c'r_1^2;$$

and when $b=0$, $c'=0$, $F=\infty$, but $F'c=0$, $\Omega=\pi$, $L=0$.

When $f=\frac{1}{3}$ (*Phil. Trans.*, 1904, p. 261),

$$\cos\psi=\frac{p-1}{p+1}, \quad \cos\psi'=\frac{p-1}{2}, \quad \mathrm{zn}\,2fF'=\frac{-p+3}{6}\sqrt{p}\,(3>p>1),$$

$$\sec\psi-\sec\psi'=1, \text{ equivalent to } \sin\chi+\cos\theta=1.$$

When $f=\frac{1}{4}$ (*Phil. Trans.*, p. 276),

$$\cos^2\psi=\frac{c}{1+c}, \quad \cos^2\psi'=c, \quad \mathrm{zn}\,2fF'=\tfrac{1}{2}(1-c), \quad \tan\psi=\sec\psi'.$$

When $f=\frac{1}{6}$ (*Trans. A. M. S.*, p. 522),

$$\sin\psi=\frac{2}{p+1}, \quad \sin^2\psi'=\frac{(p+1)(-p+3)}{4p}, \quad \mathrm{zn}\,2fF'=\frac{-p^2+9}{12\sqrt{p}},$$

$$\tan^2\psi-\tan^2\psi'=\frac{1-\sin\psi}{1+\sin\psi}, \quad \tan^2\psi'=\frac{2\sin\psi-1}{\cos^2\psi}.$$

LONDON, 1 Staple Inn, W. C., *December* 13, 1910.